Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials

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[1]

Abstract

We investigate the construction of diffusions consisting of infinitely numerous Brownian particles moving in $\mathbb{R}^d$ and interacting via logarithmic functions (2D Coulomb potentials). These potentials are really strong and long range in nature. The associated equilibrium states are no longer Gibbs measures.

We present general results for the construction of such diffusions and, as applications thereof, construct two typical interacting Brownian motions with logarithmic interaction potentials, namely the Dyson model in infinite dimensions and Ginibre interacting Brownian motions. The former is a particle system in $\mathbb{R}$ while the latter is in $\mathbb{R}^2$. Both models are translation and rotation invariant in space, and as such, are prototypes of dimensions $d = 1, 2$, respectively. The equilibrium states of the former diffusion model are determinantal random point fields with sine kernels. They appear in the thermodynamical limits of the spectrum of the ensembles of Gaussian random matrices such as GOE, GUE and GSE. The equilibrium states of the latter diffusion model are the thermodynamical limits of the spectrum of the ensemble of complex non-Hermitian Gaussian random matrices known as the Ginibre ensemble.

1 Introduction

Interacting Brownian motions (IBMs) in infinite dimensions are diffusions $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{Z}}$ consisting of infinitely many particles moving in $\mathbb{R}^d$ with the effect of the external force coming from a self potential $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and that of the mutual interaction coming from an interacting potential $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that $\Psi(x, y) = \Psi(y, x)$.

@ Intuitively, an IBM is described by the infinitely dimensional stochastic differential equation (SDE) of the form

$$dX_t^i = dB_t^i - \frac{1}{2}\nabla\Phi(X_t^i)dt - \frac{1}{2}\sum_{j\in\mathbb{Z}, j\neq i} \nabla\Psi(X_t^i, X_t^j)dt \quad (i \in \mathbb{Z}). \tag{1.1}$$

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The state space of the process $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{Z}}$ is $(\mathbb{R}^d)^{\mathbb{Z}}$ by construction. Let X be the configuration valued process given by

$$\mathsf{X}_t = \sum_{i\in\mathbb{Z}} \delta_{X_t^i}. \tag{1.2}$$

Here δ_a denotes the delta measure at a and a configuration is a Radon measure consisting of a sum of delta measures. We call $\mathbf{X}$ the labelled dynamics and X the unlabelled dynamics.

The SDE (1.1) was initiated by Lang [10], [11]. He studied the case $\Phi = 0$, and $\Psi(x,y) = \Psi(x-y)$, where Ψ is of $C_0^3(\mathbb{R}^d)$, superstable and regular according to Ruelle [20]. With the last two assumptions, the corresponding unlabelled dynamics X has Gibbsian equilibrium states. See [21], [5], and [24] for other works concerning the SDE (1.1).

In [14] the unlabelled diffusion was constructed using the Dirichlet form approach. The advantage of this method is that it gives a general and simple proof of construction, and, more significantly, allows us to apply singular interaction potentials, which are particularly of interestss, such as the Lennard-Jones 6-12 potential, hard core potential and so on. We remark that all these potentials were excluded by the SDE approach. See [27], [1] [25], and [26] for other works concerning the Dirichlet form approach to IBMs.

We remark that in all these works, except some parts of [14], the equilibrium states are supposed to be Gibbs measures with Ruelle's class interaction potentials Ψ. Thus the equilibrium states are described by the DLR equations (see (1.3)), the usage of which plays a pivotal role in the previous works.

The purpose of this paper is to construct unlabelled IBMs in infinite dimensions with the logarithmic interaction potentials

$$\Psi(x,y) = -\beta \log|x-y|. \tag{1.3}$$

We present a sequence of general theorems to construct IBMs and apply these to logarithmic potentials. We remark that the equilibrium states are not Gibbs measures because the logarithmic interaction potentials are unbounded at infinity.

The above potential Ψ in (1.3) is known to be the two-dimensional Coulomb potential. In practice, such systems are regarded as one-component plasma consisting of equally charged particles. To prevent the particles all repelling to explode, a neutralizing background charge is imposed. The self potential Φ denotes this particle-background interaction (see [3]).

We study two typical examples, namely Dyson's model (Section 2.1) and Ginibre IBMs (Section 2.2). In the first example, we take $d=1$, $\Phi=0$, and $\Psi(x,y) = -\beta\log|x-y|$ ($\beta = 1,2,4$), while in the second $d=2$, $\Phi(z) = |z|^2$, and $\Psi(x,y) = -2\log|x-y|$.

For the special values $\beta = 1,2,4$ and particular self potentials Φ the associated equilibrium states are limits of the spectrum of random matrices. Recently, much intensive research has been carried out on random point fields related to random matrices. Our purpose in this paper is a rather more dynamical one; that is, we construct diffusions, the equilibrium states of which are these random point fields related to random matrices.

The labeled dynamics of the Dyson model in infinite dimensions is represented by the following SDE.

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{R\to\infty} \sum_{|X_t^j|\le R,\, j\in\mathbb{Z},\, j\ne i} \frac{1}{X_t^i - X_t^j}\,dt \quad (i\in\mathbb{Z}). \tag{1.4}$$

Here $\beta = 1, 2, 4$, corresponding to the Gaussian orthogonal ensemble (GOE), the Gaussian unitary ensemble (GUE) and the Gaussian symplectic ensemble (GSE), respectively. The invariant probability measures $\mu_{\mathrm{dys},\beta}$ of the (unlabeled) Dyson models are translation invariant. Hence, if the distribution of X_0 equals $\mu_{\mathrm{dys},\beta}$, then for all t

$$\sum_{j\in\mathbb{Z}, j\neq i} \frac{1}{|X_t^i - X_t^j|} dt = \infty \quad \text{a.s.}. \tag{1.5}$$

This means that only a conditional convergence is possible in the summation of the drift term in (1.4), which is the cause of the difficulty in dealing with the Dyson model. It is well known that the equilibrium states are the thermodynamic limits of the distribution of the spectrum of Gaussian random matrices at the bulk [22], [3], [13].

The labeled dynamics of Ginibre IBMs is represented by the following SDE. For convenience we regard S as $\mathbb{C}$ rather than $\mathbb{R}^2$.

$$dZ_t^i = dB_t^i - Z_t^i dt + \lim_{R\to\infty} \sum_{|Z_t^j|\le R,\ j\in\mathbb{Z},\ j\neq i} \frac{Z_t^i - Z_t^j}{|Z_t^i - Z_t^j|^2} dt \quad (i \in \mathbb{Z}). \tag{1.6}$$

Here $Z_t^i = X_t^i + \sqrt{-1}Y_t^i \in \mathbb{C}$ and $\{B_t^i\}_{i\in\mathbb{Z}}$ are independent complex Brownian motions. That is, $B_t^i = B_t^{i,\mathrm{Re}} + \sqrt{-1}B_t^{i,\mathrm{Im}}$, where $\{B_t^{i,\mathrm{Re}}, \sqrt{-1}B_t^{i,\mathrm{Im}}\}_{i\in\mathbb{Z}}$ is a system of independent 1-dimensional Brownian motions. The stationary measure μ_{gin} of the unlabeled dynamics is the thermodynamic limit of the distribution of the spectrum of random Gaussian matrices called the Ginibre ensemble (cf. [22]). μ_{gin} is a random point field with logarithmic interaction potential and is known to be translation invariant. If Ginibre IBMs $\mathsf{Z} = \{\mathsf{Z}_t\} = \{\sum_i \delta_{Z_t^i}\}$ start from the stationary measure μ_{gin}, then Z is also translation invariant in space. Moreover Ginibre IBMs Z satisfy the SDE of the translation invariant form:

$$dZ_t^i = dB_t^i + \lim_{R\to\infty} \sum_{|Z_t^i - Z_t^j|\le R,\ j\in\mathbb{Z},\ j\neq i} \frac{Z_t^i - Z_t^j}{|Z_t^i - Z_t^j|^2} dt \quad (i \in \mathbb{Z}). \tag{1.7}$$

This variety of SDE representations of Ginibre IBMs is a result of the strength of the interaction potential.

A diffusion (X, P) is a family of probability measures $\mathsf{P} = \{\mathsf{P}_{\mathsf{x}}\}$ with continuous sample path $\mathsf{X} = \{\mathsf{X}_t\}$ starting at each point x of the state space with a strong Markov property (see [4]). We emphasize that we construct not only a Markov semi-group or a stationary Markov process but also a diffusion in the above sense, and also that, to apply stochastic analysis effectively, we require the construction of diffusions.

In a forth coming paper we give another general result of the SDE representation of unlabeled diffusions constructed in this paper. SDEs (1.4), (1.6) and (1.7) of the labeled dynamics are solved there.

Because of the long range nature of the logarithmic interaction, construction of the diffusion has not yet been done. The only exception is the Dyson model with $\beta = 2$. In [23] Spohn proved the closability of the Dirichlet form associated with (1.1) for this model. This implies the construction of the unlabeled dynamics (1.2) in the sense of an L^2-Markovian semigroup. An associated diffusion was constructed in [14] by combining Spohn's result with the result from [14, Theorem 0.1] on the quasi-regularity of Dirichlet forms.

In a one space dimension, some explicit computations of space-time correlation functions of infinite particle systems related to random matrices have been obtained. Indeed, Katori

and Tanemura [9] recently studied the thermodynamic limit of the space-time correlation functions related to the Dyson model and Airy process. Their limit space-time correlation functions define a stochastic process started from a limited set of initial distributions. However, the Markov (semi-group) property of the process has not yet been proved. They also proved that, if their process is Markovian, the associated Dirichlet form is the same as the one obtained in this paper and their processes coincide with the processes constructed here. It is an interesting open problem to prove the Markov property of their processes and the identification of these two processes. We also refer to [6], [7], [8], and [18] for stochastic processes of one dimensional infinite particle systems related to random matrices.

As for two dimensional infinite systems with logarithmic interactions, the construction of stochastic processes based on the explicit computation of space-time correlation functions has not been done. Techniques useful in one dimension, such as the Karlin-McGregor formula, are no longer valid in two dimensions.

Let us briefly explain the main idea. We introduce the notion of quasi Gibbs measures as a substitution for Gibbs measures. These measures satisfy inequality (2.8) involving a (finite volume) Hamiltonian. Inequality (2.8) is sufficient for the closability of the Dirichlet forms and the construction of the diffusions.

To obtain the above mentioned inequality we control the difference of the infinite volume Hamiltonians in stead of the Hamiltonian itself. The key point of the control is the usage of the geometric property of the random point fields behined the dynamics. Indeed, although the difference still diverges for Poisson random fields and Gibbs measures with translation invariance, it becomes finite for random point fields such as the Dyson random point fields and the Ginibre random point fields. For these random point fields the fluctuations of particles are extremely suppressed because the logarithmic potentials are quite strong. This cancels the sum of the difference of the infinite volume Hamiltonians.

The organization of the paper is as follows. In Section 2 we describe the set up and state the main results. We first give a set of general results (Theorems 2.2, 2.3, 2.4). Then, as applications, we construct the diffusions of the Dyson model and the Ginibre IBMs cited above in Theorem 2.5 and Theorem 2.6, respectively. Section 3 is devoted to preparation from the Dirichlet form theory and we prove Proposition 2.1. In Section 4 we prove Theorem 2.2. In Section 5 we prove Theorem 2.3. In Section 6 we prove Theorem 2.4. In Section 7 we give a sufficient condition for (2.36) when the stationary measures μ are a determinantal random point field with translation invariant kernel. This result is used in Section 8 to apply the Dyson model. In Section 8 we prove Theorem 2.5. In Section 9 we prove Theorem 2.6. In Appendix 10.1 we give a proof of Lemma 3.4, in Appendix 10.2 we prove Lemma 4.1, in Appendix 10.3 we prove (8.31) and (8.32) and in Appendix 10.4 we prove Lemma 9.2. Lemma 9.2 is a uniform estimate of the variance of the n-particle system of the Ginibre random point field to be used in the proof of Theorem 2.6.

2 Set up and main results

Let S be a closed set in $\mathbb{R}^d$ such that $0 \in S$ and $\overline{S^{\mathrm{int}}} = S$, where S^{int} means the interia of S. Let $\tilde{S}_r = \{s \in S\,; |s| \le r\}$. Let $\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,; \mathsf{s}(\tilde{S}_r) < \infty \text{ for all } r \in \mathbb{N}\}$ be the set of configurations on S. We endow S with the vague topology, under which S is a Polish space.

Let μ be a probability measure on $(\mathsf{S}, \mathcal{B}(\mathsf{S}))$. We construct μ-reversible diffusions (X, P)

with state space S by using the Dirichlet form theory. Hence we begin by introducing Dirichlet forms in the following.

For a subset $A \subset S$ we define the map $\pi_A : \mathsf{S} \to \mathsf{S}$ by $\pi_A(\mathsf{s}) = \mathsf{s}(A \cap \cdot)$. We say a function $f : \mathsf{S} \to \mathbb{R}$ is local if f is $\sigma[\pi_A]$-measurable for some bounded Borel set A. We say f is smooth if $\tilde{f}$ is smooth, where $\tilde{f}((s_i))$ is the permutation invariant function in (s_i) such that $f(\mathsf{s}) = \tilde{f}((s_i))$ for $\mathsf{s} = \sum_i \delta_{s_i}$.

Let $\mathsf{S} \bullet S = \{(\mathsf{s}, s) \in \mathsf{S} \times S\,;\, \mathsf{s}(\{s\}) \geq 1\}$. Let $a = (a_{kl}) : \mathsf{S} \bullet S \to \mathbb{R}^{d^2}$ be such that $a_{kl} = a_{lk}$ and $(a_{kl}(\mathsf{s}, s))$ is nonnegative definite. Set

$$\mathbb{D}^a[f,g](\mathsf{s}) = \frac{1}{2} \sum_i \sum_{k,l=1}^{d} a_{kl}(\mathsf{s}, s_i) \frac{\partial \widetilde{f}}{\partial s_{ik}} \cdot \frac{\partial \widetilde{g}}{\partial s_{il}}. \tag{2.1}$$

Here $s_i = (s_{i1}, \ldots, s_{id}) \in S$ and $\mathsf{s} = \sum_i \delta_{s_i}$. For given f and g, it is easy to see that the right hand side depends only on s. So the square field $\mathbb{D}^a[f,g]$ is well defined. We assume $\mathbb{D}^a[f,g] : \mathsf{S} \to \mathbb{R}$ is $\mathcal{B}(\mathsf{S})$-measurable for each of the local, smooth functions f and g.

For a and μ we consider the bilinear form $(\mathcal{E}^{a,\mu}, \mathcal{D}_\infty^{a,\mu})$ defined by

$$\begin{aligned} \mathcal{E}^{a,\mu}(f,g) &= \int_{\mathsf{S}} \mathbb{D}^a[f,g] d\mu, \\ \mathcal{D}_\infty^{a,\mu} &= \{ f \in L^2(\mathsf{S}, \mu); f \text{ is local and smooth, } \ \mathcal{E}^{a,\mu}(f,f) < \infty \}. \end{aligned} \tag{2.2}$$

When $a_{kl} = \delta_{kl}$ (δ_{kl} is the Kronecker delta), we write $\mathbb{D}^a = \mathbb{D}$, $\mathcal{E}^{a,\mu} = \mathcal{E}^{\mu}$ and $\mathcal{D}_\infty^{a,\mu} = \mathcal{D}_\infty^{\mu}$.

All examples in this paper satisfy $a_{kl} = \delta_{kl}$. We however state the assumption in a general framework. We assume the coefficients $\{a_{kl}\}$ satisfy the following:

(A.0) There exists a nonnegative, bounded, lower semicontinuous function $a_0 : \mathsf{S} \bullet S \to [0, \infty)$ and a constant $c_1 \geq 1$ such that

$$c_1^{-1} a_0(\mathsf{s}, s)|x|^2 \leq \sum_{k,l=1}^{d} a_{kl}(\mathsf{s}, s) x_k x_l \leq c_1 a_0(\mathsf{s}, s)|x|^2 \tag{2.3}$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $(\mathsf{s}, s) \in \mathsf{S} \bullet S$.

We call a function ρ^n the n-correlation function of μ (with respect to the Lebesgue measure) if $\rho^n : S^n \to \mathbb{R}$ is the permutation invariant function such that

$$\int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(x_1, \ldots, x_n) dx_1 \cdots dx_n = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i) - k_i)!} d\mu \tag{2.4}$$

for any sequence of disjoint bounded measurable subsets $A_1, \ldots, A_m \subset S$ and a sequence of natural numbers $k_1, \ldots, k_m$ satisfying $k_1 + \cdots + k_m = n$. It is well known [22] that under a mild condition the correlation functions $\{\rho^n\}_{n \in \mathbb{N}}$ determine the measure μ.

We assume μ satisfies the following.

(A.1) The measure μ has a locally bounded, n-correlation function ρ^n for each $n \in \mathbb{N}$.

We introduce a Hamiltonian on a bounded Borel set A as follows. For Borel measurable functions $\Phi : S \to \mathbb{R} \cup \{\infty\}$ and $\Psi : S \times S \to \mathbb{R} \cup \{\infty\}$ with $\Psi(x,y) = \Psi(y,x)$ let

$$\mathcal{H}_A^{\Phi,\Psi}(\mathsf{x}) = \sum_{x_i \in A} \Phi(x_i) + \sum_{x_i, x_j \in A, i<j} \Psi(x_i, x_j), \quad \text{ where } \mathsf{x} = \sum_i \delta_{x_i}. \tag{2.5}$$

We assume $\Phi < \infty$ a.e. to avoid triviality.

For two measures ν_1, ν_2 on a measurable space $(\Omega, \mathcal{B})$ we write $\nu_1 \le \nu_2$ if $\nu_1(A) \le \nu_2(A)$ for all $A \in \mathcal{B}$. We say a sequence of finite Radon measures $\{\nu^N\}$ on a Polish space Ω converge weakly to a finite Radon measure ν if $\lim_{N\to\infty} \int f d\nu^N = \int f d\nu$ for all $f \in C_b(\Omega)$.

Throughout this paper $\{b_r\}$ denotes an increasing sequence of natural numbers. We set

$$S_r = \{s \in S\,;\, |s| \le b_r\}, \quad \mathsf{S}_r^m = \{\mathsf{s} \in \mathsf{S}; \mathsf{s}(S_r) = m\}, \quad \mathcal{H}_r(\mathsf{x}) = \mathcal{H}_{S_r}^{\Phi,\Psi}(\mathsf{x}). \tag{2.6}$$

Definition 2.1. A probability measure μ is said to be a (Φ, Ψ)-quasi Gibbs measure if there exist an increasing sequence $\{b_r\}$ of natural numbers and measures $\{\mu_{r,k}^m\}$ such that, for each $r, m \in \mathbb{N}$, $\mu_{r,k}^m$ and $\mu_r^m := \mu(\cdot \cap \mathsf{S}_r^m)$ satisfy

$$\mu_{r,k}^m \le \mu_{r,k+1}^m \text{ for all } k, \quad \lim_{k\to\infty} \mu_{r,k}^m = \mu_r^m \quad \text{weakly,} \tag{2.7}$$

and that, for all $r, m, k \in \mathbb{N}$ and for $\mu_{r,k}^m$-a.e. $\mathsf{s} \in \mathsf{S}$

$$c_2^{-1} e^{-\mathcal{H}_r(\mathsf{x})} 1_{\mathsf{S}_r^m}(\mathsf{x}) \Lambda(d\mathsf{x}) \le \mu_{r,k,\mathsf{s}}^m(d\mathsf{x}) \le c_2 e^{-\mathcal{H}_r(\mathsf{x})} 1_{\mathsf{S}_r^m}(\mathsf{x}) \Lambda(d\mathsf{x}). \tag{2.8}$$

Here $c_2 = c_2(r, m, k, \pi_{S_r^c}(\mathsf{s}))$ is a positive constant, Λ is the Poisson random point field whose intensity is the Lebesgue measure on S, and $\mu_{r,k,\mathsf{s}}^m$ is the conditional probability measure of $\mu_{r,k}^m$ defined by

$$\mu_{r,k,\mathsf{s}}^m(d\mathsf{x}) = \mu_{r,k}^m(\pi_{S_r} \in d\mathsf{x} |\ \pi_{S_r^c}(\mathsf{s})). \tag{2.9}$$

(A.2) μ is a (Φ, Ψ)-quasi Gibbs measure.

Remark 2.1. (1) By definition $\mu_{r,k}^m((\mathsf{S}_r^m)^c) = 0$. Since $\mu_{r,k,\mathsf{s}}^m$ is $\sigma[\pi_{S_r^c}]$-measurable in s, we have the disintegration of the measure $\mu_{r,k}^m$

$$\mu_{r,k}^m \circ \pi_{S_r}^{-1}(d\mathsf{x}) = \int_{\mathsf{S}} \mu_{r,k,\mathsf{s}}^m(d\mathsf{x}) \mu_{r,k}^m(d\mathsf{s}). \tag{2.10}$$

(2) Let $\mu_{r,\mathsf{s}}^m(d\mathsf{x}) = \mu_r^m(\pi_{S_r}(\mathsf{s}) \in d\mathsf{x} |\ \pi_{S_r^c}(\mathsf{s}))$. Recall that a probability measure μ is said to be a (Φ, Ψ)-canonical Gibbs measure if μ satisfies the DLR equation (2.11), that is, for all $r, m \in \mathbb{N}$, the conditional probability $\mu_{r,\mathsf{s}}^m$ is given by

$$\mu_{r,\mathsf{s}}^m(d\mathsf{x}) = \frac{1}{c_3} e^{-\mathcal{H}_r(\mathsf{x}) - \mathcal{H}_{S_r,\mathsf{s}}^{\Psi}(\mathsf{x})} 1_{\mathsf{S}_r^m}(\mathsf{x}) \Lambda(d\mathsf{x}) \quad \text{for } \mu_r^m\text{-a.e. } \mathsf{s}. \tag{2.11}$$

Here $0 < c_3 < \infty$ is the normalization and $\mathcal{H}_{S_r,\mathsf{s}}^{\Psi}(\mathsf{x}) = \sum_{x_i \in S_r, s_j \in S_r^c} \Psi(x_i, s_j)$, where $\mathsf{x} = \sum_i \delta_{x_i}$ and $\mathsf{s} = \sum_j \delta_{s_j}$.

We remark that (Φ, Ψ)-canonical Gibbs measures are (Φ, Ψ)-quasi Gibbs measures. The converse is however not true. When $\Psi(x, y) = -\beta \log |x - y|$ and μ are translation invariant, μ are not (Φ, Ψ)-canonical Gibbs measure. This is because the DLR equation does not make sense. Indeed, $|\mathcal{H}_{S_r,\mathsf{s}}^{\Psi}(\mathsf{x})| = \infty$ for μ-a.s. s. The point is that one can expect a cancellation between c_3 and $e^{-\mathcal{H}_{S_r,\mathsf{s}}^{\Psi}(\mathsf{x})}$ even if $|\mathcal{H}_{S_r,\mathsf{s}}^{\Psi}(\mathsf{x})| = \infty$.

(A.3) There exist upper semicontinuous functions $\Phi_0, \Psi_0 : S \to \mathbb{R} \cup \{\infty\}$ and positive constants c_4 and c_5 such that

$$c_4^{-1} \Phi_0(s) \le \Phi(s) \le c_4 \Phi_0(s) \tag{2.12}$$

$$c_5^{-1} \Psi_0(s - t) \le \Psi(s, t) \le c_5 \Psi_0(s - t), \quad \Psi_0(s) = \Psi_0(-s) \quad (\forall s). \tag{2.13}$$

Moreover, Φ_0 and Ψ_0 are locally bounded from below and $\Gamma := \{s\,;\, \Psi_0(s) = \infty\}$ is a compact set.

We use the following result obtained in [14] and [15].

Proposition 2.1 ([14], [15]). *Assume* (A.0)–(A.3). *Then* $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}_{\infty}, L^2(\mathsf{S},\mu))$ *is closable, and its closure* $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$ *is a local, quasi-regular Dirichlet space.*

See Section 3 for the definition of "a local, quasi-regular Dirichlet space" and necessary notions of the Dirichlet form theory. Combining Proposition 2.1 with the Dirichlet form theory developed in [4] and [12], we obtain the following.

Corollary 2.1 . Assume (A.0)–(A.3). Then there exists a diffusion (X,P) associated with $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$. Moreover, the diffusion (X,P) is μ-reversible.

We say a diffusion (X,P) is associated with the Dirichlet space $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$ if $E_{\mathsf{x}}[f(\mathsf{X}_t)] = T_t f(\mathsf{x})$ μ-a.e. x for all $f \in L^2(\mathsf{S},\mu)$. Here T_t is the L^2-semi group associated with the Dirichlet space $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$. Moreover, (X,P) is called μ-reversible if (X,P) is μ-symmetric and μ is an invariant probability measure of (X,P).

Assumptions (A.0), (A.1), and (A.3) are easily verified. The most crucial assumption in Proposition 2.1 is (A.2). To obtain a sufficient condition for (A.2) we introduce assumptions (A.4) and (A.5) below. We assume μ has a good finite particle approximation $\{\mu^N\}_{N\in\mathbb{N}}$ in the following sense.

(A.4) There exists a sequence of probability measures $\{\mu^N\}_{N\in\mathbb{N}}$ on S such that $\{\mu^N\}_{N\in\mathbb{N}}$ have the n-correlation functions $\{\rho^n_N\}_{N\in\mathbb{N}}$ satisfying, for all $n \in \mathbb{N}$,

$$\lim_{N\to\infty} \rho^n_N(x_1,\dots,x_n) = \rho^n(x_1,\dots,x_n) \quad \text{a.e.,} \tag{2.14}$$

$$\sup_N \sup_{\tilde{S}^n_r} |\rho^n_N(x_1,\dots,x_n)| \le \{c_6 n^\delta\}^n, \tag{2.15}$$

where $c_6 > 0$ and $\delta < 1$ are constants depending on $r \in \mathbb{N}$. Moreover, $\mu^N(\mathsf{s}(S) \le n_N) = 1$ for some $n_N \in \mathbb{N}$ and μ^N is a (Φ^N, Ψ^N)-canonical Gibbs measure, that is, μ^N satisfies (2.11) for all $r, m \in \mathbb{N}$. In addition, the potentials $\Phi^N : S \to \mathbb{R}\cup\{\infty\}$ and $\Psi^N : S\times S \to \mathbb{R}\cup\{\infty\}$ satisfy $\Psi^N(x,y) = \Psi^N(y,x)$ and the following:

$$\lim_{N\to\infty} \Phi^N(x) = \Phi(x) \quad \text{for a.e. } x, \qquad \inf_{N, x\in S} \Phi^N(x) > -\infty. \tag{2.16}$$

$$\begin{aligned} &\Psi^N \in C^1(S\times S\backslash\{x=y\}) \\ &\lim_{N\to\infty} \Psi^N(x,y) = \Psi(x,y) \quad \text{compact uniformly in } C^1(S\times S\backslash\{x=y\}) \\ &\inf_{N\in\mathbb{N}} \inf_{x,y\in S_r} \Psi^N(x,y) > -\infty \quad \text{for all } r\in\mathbb{N}. \end{aligned} \tag{2.17}$$

Remark 2.2. By (2.14) and (2.15) we see that $\lim_{N\to\infty}\mu^N = \mu$ weakly in S (see Lemma 4.1). By $\mu^N(\mathsf{s}(S) \le n_N) = 1$, (2.11) makes sense even if Ψ^N is a logarithmic function. Moreover, by (2.17) $\Psi \in C^1(S\times S\backslash\{x=y\})$. So we take Γ in (A.3) as $\Gamma = \{0\}$ or $\emptyset$.

The difficulty in treating the logarithmic interaction is the unboundedness at infinity. Indeed, the DLR equation does not make sense for infinite volume. The key issue in overcoming this difficulty is the fact that the logarithmic functions have small variations at infinity. With this property we can control the difference of interactions rather than the interactions themselves. Bearing this in mind we introduce the quantity (2.20) and assumption (A.5) below.

We consider sequences of bounded sets $\{S_r^N\}_{r\in\mathbb{N}}$ satisfying

$$\bigcup_{r=1}^{\infty} S_r^N = S \text{ for all } N\in\mathbb{N}, \quad S_r^N \subset S_s^N \text{ for all } r<s\in\mathbb{N}, \tag{2.18}$$

$$\lim_{N\to\infty} d_{\mathrm{Haus}}(S_r^N, S_r) = 0 \quad \text{for all } r\in\mathbb{N}. \tag{2.19}$$

Here $d_{\mathrm{Haus}}(A,B) := \inf\{\varepsilon>0;\ A^\varepsilon \supset B,\ B^\varepsilon \supset A\}$ is the Hausdorff distance of sets, where A^ε is the ε neighborhood of A. For $\{S_r^N\}$ as above we set $S_{rs}^N = S_s^N \backslash S_r^N$. We define the map $\mathsf{h}_{r,s}^N : S_r^N \times \mathsf{S} \to \mathbb{R}\cup\{\infty\}$ and the set $\mathsf{H}_{r,k}$ as

$$\mathsf{h}_{r,s}^N(x,\mathsf{y}) = \sum_{y_i\in S_{rs}^N} \{\Psi^N(x,y_i) - \Psi^N(0,y_i)\} \qquad (\mathsf{y}=\sum_i \delta_{y_i}) \tag{2.20}$$

$$\mathsf{H}_{r,k} = \{\mathsf{y}\in\mathsf{S}\,;\ \sup_{N\in\mathbb{N}} \sup_{r<s\in\mathbb{N}} \sup_{x\neq x'\in S_r^N} \frac{|\mathsf{h}_{r,s}^N(x,\mathsf{y}) - \mathsf{h}_{r,s}^N(x',\mathsf{y})|}{|x-x'|} \le k\}. \tag{2.21}$$

We now assume the following.
(A.5) There exist increasing sequences of Borel sets $\{S_r^N\}$ satisfying (2.18), (2.19) and

$$\lim_{k\to\infty} \liminf_{N\to\infty} \mu^N(\mathsf{H}_{r,k}) = 1 \quad \text{for all } r\in\mathbb{N}. \tag{2.22}$$

Theorem 2.2. *Assume* (A.4) *and* (A.5). *Then μ is a (Φ,Ψ)-quasi Gibbs measure.*

Corollary 2.2 . Assume (A.0), (A.1) and (A.3)–(A.5). Then we have the following.
(1) $(\mathcal{E}^{a,\mu}, \mathcal{D}_\infty^{a,\mu}, L^2(\mathsf{S},\mu))$ is closable, and its closure $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$ is a local, quasi-regular Dirichlet space.
(2) There exists a μ-reversible diffusion (X,P) associated with $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$.

We give a sufficient condition for (A.5) when Ψ is a logarithmic function. We assume:
(A.6) Ψ^N is of the form

$$\Psi^N(x,y) = -\beta \log|\varpi_N(x) - \varpi_N(y)|. \tag{2.23}$$

Here $\beta\in(0,\infty)$ is a constant and $\varpi_N = (\varpi_N^1, \varpi_N^2) \in C^\infty(S;\mathbb{R}^{2d})$ are maps such that

$$\lim_{N\to\infty} \|\varpi_N - (x,0)\|_{C^n(|x|\le R)} = 0 \quad \text{for all } R,n\in\mathbb{N}. \tag{2.24}$$

Here $\|\cdot\|_{C^n(|x|\le R)}$ is the C^n-norm on the set $\{x\in S; |x|\le R\}$.
Remark 2.3. (1) By (2.17) and (A.6) we see that $\Psi(x,y) = -\beta\log|x-y|$.
(2) We take ϖ_N to be $\mathbb{R}^{2d}$-valued rather than $\mathbb{R}^d$-valued in (A.6) because we use a circular ensemble as a finite particle approximation μ^N for the Dyson model in Theorem 2.5. In this case we approximate $\mathbb{R}$ by a torus embedded in the upper half plane $\mathbb{R}_+^2 = \{(x,y)\,;\,y\ge 0\}$.
(3) As in the Ginibre random point field (Theorem 2.6) we simply take $\varpi_N(x) = (x,0)$, that is $\varpi_N^2 = 0$. Thus for a Ginibre random point field we do not extend the range of ϖ_N to $\mathbb{R}^4$ from $\mathbb{R}^2$. In this case $\Psi^N(x,y) = \Psi(x,y) = -2\log|x-y|$ for all N.

We denote $\langle \mathsf{s}, f\rangle = \sum_i f(s_i)$ for $\mathsf{s} = \sum_i \delta_{s_i}$. Then by construction

$$\mathsf{h}_{r,s}^N(x,\mathsf{s}) = \langle \mathsf{s}, 1_{S_{rs}^N}\{\Psi^N(x,\cdot) - \Psi^N(0,\cdot)\}\rangle. \tag{2.25}$$

Let $\mathbf{e}_m = (\delta_{mn})_{1\le n\le 2d}$, where $m=1,\dots,2d$. Let $\mathbf{R}_m$ denote the $2m$-dimensional subspace of $\mathbb{R}^{2d}$ spanned by $\{\mathbf{e}_i, \mathbf{e}_{d+i}\}_{1\le i\le m}$. Let $\mathbf{P}_m$ denote the orthogonal projection onto

the subspace $\mathbf{R}_m$ under the standard inner product on $\mathbb{R}^{2d}$. For $2 \le m \le d$ and a nonzero vector $x \in \mathbb{R}^{2d}$ let $\theta_m(x)$ denote the angle between the vector $\mathbf{P}_m(x)$ and the subspace $\mathbf{R}_{m-1}$. We take $\theta_m(x)$ to be $-\pi/2 \le \theta_m(x) \le \pi/2$. For $m = 1$ and a nonzero vector $x \in \mathbb{R}^{2d}$ let $\theta_1(x)$ be the angle between $\mathbf{P}_1(x)$ and $\mathbf{e}_1$. We take $-\pi < \theta_1(x) \le \pi$, $\theta_1(\mathbf{e}_1) = 0$ and $\theta_1(\mathbf{e}_{d+1}) = \pi/2$. (Recall that $\mathbf{P}_1(x) \in \mathbf{R}_1$ and that $\mathbf{R}_1$ is spanned by $\mathbf{e}_1$ and $\mathbf{e}_{d+1}$).

Let $\mathsf{t}_1(\theta) = \cos(\theta)$ and $\mathsf{t}_{-1}(\theta) = \sin(\theta)$. Let $\mathbb{I} = \{\mathbf{i} = (i_1, \dots, i_d)\,;\, i_j = \pm 1\}$. For $\mathbf{i} \in \mathbb{I}$ and $\ell \in \mathbb{N}$, let $\mathsf{t}_{\mathbf{i},\ell} : \mathbb{R}^{2d} \backslash \{0\} \to \mathbb{R}$ be the function defined by

$$\mathsf{t}_{\mathbf{i},\ell}(x) = \prod_{m=1}^{d} \mathsf{t}_{i_m}(\ell\, \theta_m(x)). \tag{2.26}$$

For an increasing sequence $\{b_r\}$ of natural numbers we take S_r^N in (A.5) as follows:

$$S_r^N = \{s \in S\,;\;\; |\varpi_N(s)| \le b_r\}. \tag{2.27}$$

We now define the map $\mathsf{u}^N_{\ell,\mathbf{i},r,s} : \mathsf{S} \to \mathbb{R}$ and the set $\mathsf{U}_{\ell,\mathbf{i},r,k}$ by

$$\mathsf{u}^N_{\ell,\mathbf{i},r,s}(\mathsf{y}) = \langle \mathsf{y}, (\mathsf{t}_{\mathbf{i},\ell} \circ \varpi_N) |\varpi_N|^{-\ell} 1_{S^N_{rs}} \rangle, \tag{2.28}$$

$$\mathsf{U}_{\ell,\mathbf{i},r,k} = \{\mathsf{y} \in \mathsf{S}\,;\, \sup_{N \in \mathbb{N}} \sup_{r < s \in \mathbb{N}} |\mathsf{u}^N_{\ell,\mathbf{i},r,s}(\mathsf{y})| \le k\}, \tag{2.29}$$

where $\mathbf{i} \in \mathbb{I}$ and $\ell, r, s \in \mathbb{N}$ such that $r < s$. Let $S^N_{1\infty} = \cup_{s=1}^{\infty} S^N_{1s}$. Let

$$\bar{\mathsf{u}}^N_{\ell}(\mathsf{y}) = \langle \mathsf{y}, |\varpi_N|^{-\ell} 1_{S^N_{1\infty}} \rangle, \tag{2.30}$$

$$\bar{\mathsf{U}}_{\ell,k} = \{\mathsf{y} \in \mathsf{S}\,;\, \sup_{N \in \mathbb{N}} \bar{\mathsf{u}}^N_{\ell}(\mathsf{y}) \le k\}. \tag{2.31}$$

With this preparation we introduce the assumption below.
(A.7) There exist $\{b_r\}$ and a natural number ℓ_0 such that

$$\lim_{k\to\infty} \liminf_{N\to\infty} \mu^N(\bar{\mathsf{U}}_{\ell_0,k}) = 1 \tag{2.32}$$

$$\lim_{k\to\infty} \liminf_{N\to\infty} \mu^N(\mathsf{U}_{\ell,\mathbf{i},r,k}) = 1 \quad \text{for all } r \in \mathbb{N},\ 1 \le \ell < \ell_0,\ \mathbf{i} \in \mathbb{I}. \tag{2.33}$$

When $\ell_0 = 1$, according to our interpretation, (2.33) always holds by convention.

We give a sufficient condition of (A.5) when the interaction is logarithmic.

Theorem 2.3. *Assume* (A.4), (A.6), *and* (A.7). *Then* (A.5) *holds.*

Condition (2.32) of (A.7) is easily checked because it follows from the estimate of the 1-correlation function. On the other hand, condition (2.33) of (A.7) is much more subtle. Hence we give a sufficient condition for (2.33). This condition is used in the proof of Theorem 2.5 and Theorem 2.6.

Let $\tilde{S}^N_r = \{s \in S;\ |\varpi_N(s)| \le r\}$ and $\tilde{S}^N_{1r} = \tilde{S}^N_r \backslash \tilde{S}^N_1$. Let $\tilde{\mathsf{u}}^{N,j}_{\ell,\mathbf{i},r} : \mathsf{S} \to \mathbb{R}$ such that

$$\tilde{\mathsf{u}}^{N,j}_{\ell,\mathbf{i},r}(\mathsf{y}) := \langle \mathsf{y}, \frac{\lceil |\varpi_N| \rceil^j}{|\varpi_N|^{\ell}} \mathsf{t}_{\mathbf{i},\ell}(\varpi_N) 1_{\tilde{S}^N_{1r}} \rangle = \sum_{y_i \in \tilde{S}^N_{1r}} \frac{\lceil |\varpi_N(y_i)| \rceil^j}{|\varpi_N(y_i)|^{\ell}} \mathsf{t}_{\mathbf{i},\ell}(\varpi_N(y_i)). \tag{2.34}$$

Here $\lceil \cdot \rceil$ is the minimal integer greater than or equal to $\cdot$ and $\mathsf{y} = \sum_i \delta_{y_i}$. Let

$$\tilde{\mathsf{u}}^{j}_{\ell,\mathbf{i},r} = \sup_{M \in \mathbb{N}} |\tilde{\mathsf{u}}^{M,j}_{\ell,\mathbf{i},r}|. \tag{2.35}$$

Theorem 2.4. *Assume* (A.4) *and* (A.6). *Assume for each* $\mathbf{i} \in \mathbb{I}$ *and* $1 \le \ell < \ell_0$, *there exists a* $j \in \mathbb{N}$ *and a positive constant* c_7 *satisfying* $1 \le j \le \ell$ *and*

$$\lim_{r\to\infty} r^{c\,\gamma^{-j}} \sup_{N\in\mathbb{N}} \|\tilde{\mathsf{u}}^{j}_{\ell,\mathbf{i},r}\|_{L^1(\mathsf{S},\mu^N)} = 0. \tag{2.36}$$

Then μ^N *satisfies* (2.33).

2.1 Dyson model in infinite dimensions (Dyson IBMs).

Let $S = \mathbb{R}$. Let $\mu_{\mathrm{dys},\beta}$ ($\beta = 1,2,4$) be the probability measure on S whose n-correlation function $\rho^n_{\mathrm{dys},\beta}$ is given by

$$\rho^n_{\mathrm{dys},\beta}(x_1,\dots,x_n) = \det(\mathsf{K}_{\sin,\beta}(x_i - x_j))_{1\le i,j\le n}. \tag{2.37}$$

Here we take $\mathsf{K}_{\sin,2}(x) = \sin(\pi x)/\pi x$. The definition of $\mathsf{K}_{\sin,\beta}$ for $\beta = 1,4$ is given by (8.4) and (8.6). We use quaternions to denote the kernel $\mathsf{K}_{\sin,\beta}$ for $\beta = 1,4$. The precise meaning of the determinant of (2.37) for $\beta = 1,4$ is given by (8.3).

The kernel $\mathsf{K}_{\sin,2}$ is called the sine kernel. We remark that $\mathsf{K}_{\sin,2}(t) = \frac{1}{2\pi}\int_{|k|\le\pi} e^{\sqrt{-1}kt}\,dk$ and $0 \le \mathsf{K}_{\sin,2} \le \mathrm{Id}$ as an operator on $L^2(\mathbb{R})$.

Theorem 2.5. $\mu_{\mathrm{dys},\beta}$ ($\beta = 1,2,4$) *satisfy assumptions* (A.1), (A.2), *and* (A.3). *Here we take* $\Phi(x) = 0$ *and* $\Psi(x,y) = -\beta\log|x-y|$ *in* (A.2).

By Corollary 2.1 and Theorem 2.5 we obtain

Corollary 2.5 1. Let $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mathsf{S},\mu))$ be the Dirichlet space in Proposition 2.1 with $a = (\delta_{kl})$ and $\mu = \mu_{\mathrm{dys},\beta}$. Then there exists a μ-reversible diffusion (X,P) associated with $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mathsf{S},\mu))$.

Remark 2.4. (1) We write $\mathsf{X}_t = \sum_{i\in\mathbb{Z}} \delta_{X^i_t}$. Here $\mathbf{X}_t = (X^i_t)_{i\in\mathbb{Z}}$ is the associated labeled dynamics. It is known [16] that particles X^i_t never collide with each other. Moreover, the associated labeled dynamics $(X^i_t)_{i\in\mathbb{Z}}$ is a solution of the SDE

$$dX^i_t = dB^i_t + \frac{\beta}{2}\lim_{R\to\infty}\sum_{|X^j_t|\le R,\, j\in\mathbb{Z},\, j\ne i} \frac{1}{X^i_t - X^j_t}\,dt \quad (i\in\mathbb{Z}) \tag{2.38}$$

with $(X^i_0) = (x_i)$ for $\mu_{\mathrm{dys},\beta}$-a.s. $\mathsf{x} = \sum_i \delta_{x_i}$.
(2) We remark that $\mu_{\mathrm{dys},\beta}$ is translation invariant. The dynamics X_t inherits the translation invariance from the equilibrium state $\mu_{\mathrm{dys},\beta}$. Indeed, if X_t starts from the distribution $\mu_{\mathrm{dys},\beta}$, then the distribution of X_t becomes translation invariant in time and space.
(3) One can easily see that $\rho^1_{\mathrm{dys},\beta}(x) = 1$. By scaling in space we can treat the $\mu_{\mathrm{dys},\beta}$ with intensity $\rho^1_{\mathrm{dys},\beta}(x) = \bar\rho$ for any $0 < \bar\rho < \infty$.

2.2 Ginibre interacting Brownian motions.

Next we proceed with the Ginibre IBMs. For this purpose we first introduce a Ginibre random point field, which is a stationary probability measure for a Ginibre IBM.

Let the state space S of particles be $\mathbb{C}$. Let

$$\mathsf{K}_{\mathrm{gin}}(z_1,z_2) = \frac{1}{\pi}\exp\Big(-\frac{|z_1|^2}{2} - \frac{|z_2|^2}{2} + z_1\cdot\bar z_2\Big). \tag{2.39}$$

Here $z_1, z_2 \in \mathbb{C}$. Let μ_{gin} be the probability measure whose n-correlation ρ^n_{gin} is given by

$$\rho^n_{\mathrm{gin}}(z_1,\dots,z_n) = \det(\mathsf{K}_{\mathrm{gin}}(z_i,z_j))_{1\le i,j\le n}. \tag{2.40}$$

We call μ_{gin} the Ginibre random point field. It is well known [13] that μ_{gin} is the thermodynamic limit of the distribution of the spectrum of random Gaussian matrix called the Ginibre ensemble (cf. [22]), which is the ensemble of complex non-Hermitian random $N\times N$ matrices whose $2N^2$ parameters are independent Gaussian random variables with mean zero and variance 1/2.

Theorem 2.6. *μ_{gin} satisfies assumptions* (A.1), (A.2), *and* (A.3). *Here we take $\Phi(z) = |z|^2$ and $\Psi(z_1,z_2) = -2\log|z_1 - z_2|$ in* (A.2).

By Corollary 2.1 and Theorem 2.6 we obtain

Corollary 2.6 . Let $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mathsf{S},\mu))$ be the Dirichlet space in Proposition 2.1 with $a = (\delta_{kl})$ and $\mu = \mu_{\mathrm{gin}}$. Then there exists a μ-reversible diffusion (Z,P) associated with $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mathsf{S},\mu))$.

We write $\mathsf{Z}_t = \sum_{i\in\mathbb{Z}} \delta_{Z^i_t}$. We see that the associated labeled dynamics $(Z^i_t)_{i\in\mathbb{Z}}$ is a solution of the SDE

$$dZ^i_t = dB^i_t - Z^i_t dt + \lim_{R\to\infty} \sum_{|Z^j_t|\le R,\ j\in\mathbb{Z},\ j\ne i} \frac{Z^i_t - Z^j_t}{|Z^i_t - Z^j_t|^2} dt \quad (i\in\mathbb{Z}). \tag{2.41}$$

Here $Z^i_t \in \mathbb{C}$ and $\{B^i_t\}_{i\in\mathbb{Z}}$ are independent complex Brownian motions.

We remark that the kernel $\mathsf{K}_{\mathrm{gin}}$ is *not* translation invariant. The measure μ_{gin} is however rotation and translation invariant. Such an invariance is inherited by the unlabeled diffusion $\mathsf{Z}_t = \sum_{i\in\mathbb{Z}} \delta_{Z^i_t}$. This may be surprising because SDE (2.41) is not translation invariant at first glance. In a forth coming paper we show that $(Z^i_t)_{i\in\mathbb{Z}}$ satisfies the following SDE

$$dZ^i_t = dB^i_t + \lim_{R\to\infty} \sum_{|Z^i_t - Z^j_t|\le R,\ j\in\mathbb{Z},\ j\ne i} \frac{Z^i_t - Z^j_t}{|Z^i_t - Z^j_t|^2} dt \quad (i\in\mathbb{Z}) \tag{2.42}$$

if Z_t starts from the distribution μ_{gin}. The passage from (2.41) to (2.42) is a result of the cancellation between the repulsion of the mutual interaction of the particles and the neutralizing background charge.

3 Preliminaries from the Dirichlet form theory.

In this section we prepare some results from the Dirichlet form theory and give a proof of Proposition 2.1. The proof of Proposition 2.1 is essentially the same as in [14] and [15] although the notion of quasi-Gibbs measures was not introduced in these papers and the statement was different to Proposition 2.1. For the reader's convenience we present the proof here.

We begin by recalling the definition of Dirichlet forms and related notions according to [4] and [12]. Let X be a Polish space and m be a σ-finite Borel measure on X whose topological support equals X. Let $\mathcal{F}$ be a dense subspace of $L^2(X,m)$ and $\mathcal{E}$ be a non-negative bilinear form defined on $\mathcal{F}$. We call $(\mathcal{E},\mathcal{F})$ a Dirichlet form on $L^2(X,m)$ if $(\mathcal{E},\mathcal{F})$ is closed and Markovian. Here we say $(\mathcal{E},\mathcal{F})$ is Markovian if $\bar{u} := \min\{\max\{u,0\},1\} \in \mathcal{F}$ and

$\mathcal{E}(\bar{u},\bar{u}) \le \mathcal{E}(u,u)$ for any $u \in \mathcal{F}$. The triplet $(\mathcal{E},\mathcal{F},L^2(X,m))$ is called a Dirichlet space. We say $(\mathcal{E},\mathcal{F},L^2(X,m))$ is local if $\mathcal{E}(u,v)=0$ for any $u,v\in\mathcal{F}$ with disjoint compact supports. Here a support of $u\in\mathcal{F}$ is the topological support of the signed measure udm (see [4]).

For a given Dirichlet space there exists an L^2-Markovian semi-group associated with the Dirichlet space. If the Dirichlet space satisfies the quasi-regularity explained below, then there exists a Hunt process associated with the Dirichlet space. Moreover, if the Dirichlet form is local, then the Hunt process becomes a diffusion, that is, a strong Markov process with continuous sample paths.

We say a Dirichlet space $(\mathcal{E},\mathcal{F},L^2(X,m))$ is quasi-regular if
(Q.1) There exists an increasing sequence of compact sets $\{K_n\}$ such that $\cup_n\mathcal{F}(K_n)$ is dense in $\mathcal{F}$ w.r.t. $\mathcal{E}_1^{1/2}$-norm. Here $\mathcal{F}(K_n)=\{f\in\mathcal{F}\,;\, f=0\ m\text{-a.e. on } K_n^c\}$, and $\mathcal{E}_1^{1/2}(f)=\mathcal{E}(f,f)^{1/2}+\|f\|_{L^2(E,m)}$.
(Q.2) There exists a $\mathcal{E}_1^{1/2}$-dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-quasi continuous m-version.
(Q.3) There exist a coutable set $\{u_n\}_{n\in\mathbb{N}}$ having $\mathcal{E}$-quasi continuous m-version $\tilde{u}_n$, and an exceptional set $\mathcal{N}$ such that $\{\tilde{u}_n\}_{n\in N}$ separates the points of $E\backslash\mathcal{N}$.

Lemma 3.1. (1) *Assume* (A.1). *Let* $(\mathcal{E}^{\mu},\mathcal{D}^{\mu}_{\infty})$ *be as in* (2.2) *with* $a_{kl}=\delta_{kl}$. *Assume* $(\mathcal{E}^{\mu},\mathcal{D}^{\mu}_{\infty})$ *is closable on* $L^2(\mathsf{S},\mu)$. *Then its closure* $(\mathcal{E}^{\mu},\mathcal{D}^{\mu})$ *on* $L^2(\mathsf{S},\mu)$ *is a local, quasi-regular Dirichlet form.*
(2) *In addition, assume* (A.0) *and that* $(\mathcal{E}^{a,\mu},\mathcal{D}^{a,\mu}_{\infty})$ *is closable* $L^2(\mathsf{S},\mu)$. *Then its closure* $(\mathcal{E}^{a,\mu},\mathcal{D}^{a,\mu}_{\infty})$ *on* $L^2(\mathsf{S},\mu)$ *is a local, quasi-regular Dirichlet form.*

Proof. (1) follows from [14, Theorem 1], in which we suppose that the density functions are locally bounded and $\sum_{m=1}^{\infty} m\mu(\mathsf{S}_r^m)<\infty$. We remark that these assumptions follow immediately from (A.1). We have thus obtained (1).

Let $c_8=c_1\sup|a_0(\mathsf{s},s)|$. Then by (A.0) we see that $c_8<\infty$ and

$$\mathcal{D}^{a,\mu}\supset\mathcal{D}^{\mu},\quad \mathcal{E}^{a,\mu}(f,f)\le c_8\mathcal{E}^{\mu}(f,f)\quad \text{for all } f\in\mathcal{D}^{\mu}.$$

Hence (2) follows from (1). □

We now proceed with the proof of closability. Let μ_r^m be as in Definition 2.1. We remark that $\sum\mu_r^m=\mu$ by construction. Let $\mathcal{E}_r^{m,a,\mu}$ be the bilinear form defined by

$$\mathcal{E}_r^{m,a,\mu}(f,g)=\int\mathbb{D}^a[f,g]d\mu_r^m. \tag{3.1}$$

Then we have $\mathcal{E}^{a,\mu}=\sum_{m=1}^{\infty}\mathcal{E}_r^{m,a,\mu}$ for each $r\in\mathbb{N}$, where $\mathcal{E}^{a,\mu}$ is the bilinear form given by (2.2). We now quote a result from [14].

Lemma 3.2 (Theorem 2 in [14])**.** *Assume* $(\mathcal{E}_r^{m,a,\mu},\mathcal{D}^{a,\mu}_{\infty})$ *is closable on* $L^2(\mathsf{S},\mu)$ *for all* $r,m\in\mathbb{N}$. *Then* $(\mathcal{E}^{a,\mu},\mathcal{D}^{a,\mu}_{\infty})$ *is closable on* $L^2(\mathsf{S},\mu)$.

Proof. When $b_r=r$ and the coefficient is the unit matrix, that is, $a_{kl}(\mathsf{s},s_i)=\delta_{kl}$, Lemma 3.2 was proved in Theorem 2 in [14]. The generalization to the present case is trivial. □

Let $\mu_{r,k}^m$ as in Definition 2.1. Define the bilinear form $\mathcal{E}_{r,k}^{m,a,\mu}$ by

$$\mathcal{E}_{r,k}^{m,a,\mu}(f,g)=\int\mathbb{D}^a[f,g]d\mu_{r,k}^m. \tag{3.2}$$

Lemma 3.3. *Assume* $(\mathcal{E}^{m,a,\mu}_{r,k}, \mathcal{D}^{a,\mu}_{\infty})$ *is closable on* $L^2(\mathsf{S}, \mu^m_{r,k})$ *for all* k*. Then* $(\mathcal{E}^{m,a,\mu}_{r}, \mathcal{D}^{a,\mu}_{\infty})$ *is closable on* $L^2(\mathsf{S}, \mu)$*.*

Proof. By (2.7) we have $\mu^m_{r,k} \le \mu$. This implies $(\mathcal{E}^{m,a,\mu}_{r,k}, \mathcal{D}^{a,\mu}_{\infty})$ is closable not only on $L^2(\mathsf{S}, \mu^m_{r,k})$ but also on $L^2(\mathsf{S}, \mu)$. By (2.7) the forms $\{(\mathcal{E}^{m,a,\mu}_{r,k}, \mathcal{D}^{a,\mu}_{\infty})\}$ are nondecreasing in k and converge to $(\mathcal{E}^{m,a,\mu}_{r}, \mathcal{D}^{a,\mu}_{\infty})$ as $k \to \infty$. Hence $(\mathcal{E}^{m,a,\mu}_{r}, \mathcal{D}^{a,\mu}_{\infty})$ is closable on $L^2(\mathsf{S}, \mu)$ by the monotone convergence theorem of closable bilinear forms. □

Let $\mu^m_{r,k,\mathsf{s}}$ be as in (2.9). Let $\mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}(f,g) = \int_{\mathsf{S}} \mathbb{D}^a[f,g] d\mu^m_{r,k,\mathsf{s}}$. By (2.10) and (3.1)

$$\mathcal{E}^{m,a,\mu}_{r,k}(f,g) = \int_{\mathsf{S}} \mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}(f,g) \mu^m_{r,k}(d\mathsf{s}), \tag{3.3}$$

$$\|f\|^2_{L^2(\mathsf{S}^m_r, \mu^m_{r,k})} = \int_{\mathsf{S}} \|f\|^2_{L^2(\mathsf{S}^m_r, \mu^m_{r,k,\mathsf{s}})} \mu^m_{r,k}(d\mathsf{s}). \tag{3.4}$$

Lemma 3.4. *Assume* $(\mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}, \mathcal{D}^{a,\mu}_{\infty})$ *is closable on* $L^2(\mathsf{S}^m_r, \mu^m_{r,k,\mathsf{s}})$ *for* $\mu^m_{r,k}$*-a.s.* s*. Then* $(\mathcal{E}^{m,a,\mu}_{r,k}, \mathcal{D}^{a,\mu}_{\infty})$ *is closable on* $L^2(\mathsf{S}, \mu^m_{r,k})$*.*

The proof of this lemma is the same as Theorem 4 in [14]. We give the proof of Lemma 3.4 in Appendix (see Section 10.1) for the reader's convenience.

Lemma 3.5. *Assume* (A.0)*,* (A.2)*, and* (A.3)*. Then* $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}_{\infty}, L^2(\mathsf{S}, \mu))$ *is closable.*

Proof. By (A.0), (A.2), and (A.3) and by using Lemma 3.2 in [14] we see that $(\mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}, \mathcal{D}^{a,\mu}_{\infty})$ is closable on $L^2(\mathsf{S}^m_r, \mu^m_{r,k,\mathsf{s}})$ for $\mu^m_{r,k}$-a.s. s. Combining this with Lemma 3.2, Lemma 3.3 and Lemma 3.4 we conclude Lemma 3.5. □

Proof of Proposition 2.1. Proposition 2.1 follows immediately from Lemma 3.1 and Lemma 3.5. □

Proof of Corollary 2.1. By Proposition 2.1 and [4, Theorems 4.5.1] there exists a μ-symmetric diffusion whose Dirichlet space is $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S}, \mu))$. Since $1 \in \mathcal{D}^{a,\mu}$, the diffusion is conservative, which completes the proof. □

4 Proof of Theorem 2.2

In this section we prove Theorem 2.2. Let μ and μ^N be the measures in Theorem 2.2. These measures satisfy (A.4) and (A.5) by assumption. We fix $r, m \in \mathbb{N}$ throughout this section.

Lemma 4.1. *Assume* (2.14) *and* (2.15) *in* (A.4)*. Then* $\lim_{N\to\infty} \mu^N = \mu$ *weakly.*

We give the proof of Lemma 4.1 in Appendix 10.2.

Let S_r and S^m_r be as in (2.6). Let S^N_r be as in (2.18). Set $\mathsf{S}^{N,m}_r = \{\mathsf{s} \in \mathsf{S}; \mathsf{s}(S^N_r) = m\}$.

Lemma 4.2. *Let* $\mathsf{H}_{r,k}$ *be as in* (2.21)*. Let* $\mu^{N,m}_{r,k} = \mu^N(\cdot \cap \mathsf{S}^{N,m}_r \cap \mathsf{H}_{r,k})$*. Then there exists a subsequence of* $\{\mu^{N,m}_{r,k}\}$*, denoted by the same symbol, and measures* $\{\mu^m_{r,k}\}$ *such that*

$$\lim_{N\to\infty} \mu^{N,m}_{r,k} = \mu^m_{r,k} \quad \text{weakly for all } r,k,m. \tag{4.1}$$

Moreover, the measures $\{\mu^m_{r,k}\}$ *satisfy* (2.7)*.*

Proof. By Lemma 4.1 we see that $\{\mu^N\}$ is a weak convergent sequence. This combined with $\mu_{r,k}^{N,m} \le \mu^N$ shows that $\{\mu_{r,k}^{N,m}\}$ is relatively compact for each $r,k,m \in \mathbb{N}$. Hence we can choose a convergent subsequence $\{\mu_{r,k}^{n_N(r,k),m}\}$ from any subsequence of $\{\mu_{r,k}^{N,m}\}$ for each r,k,m. Then by diagonal argument we obtain (4.1).

Since $\mathsf{H}_{r,k} \subset \mathsf{H}_{r,k+1} \subset \mathsf{S}$, we have $\mu_{r,k}^{N,m} \le \mu_{r,k+1}^{N,m}$. This deduces $\mu_{r,k}^{m} \le \mu_{r,k+1}^{m}$ by (4.1). By (4.1) we see that, for $f \in C_b(\mathsf{S})$

$$\begin{aligned} |\int f d\mu_{r,k}^m - \int f d\mu_r^m| \le & \lim_{N\to\infty} |\int f d\mu_{r,k}^m - \int f d\mu_{r,k}^{N,m}| \\ & + \limsup_{N\to\infty} |\int f d\mu_{r,k}^{N,m} - \int f d\mu_r^{N,m}| + \lim_{N\to\infty} |\int f d\mu_r^{N,m} - \int f d\mu_r^m| \\ = & \limsup_{N\to\infty} |\int f d\mu_{r,k}^{N,m} - \int f d\mu_r^{N,m}| \\ \le & \{\sup_{\mathsf{s}} |f(\mathsf{s})|\} \cdot \limsup_{N\to\infty} \mu_r^{N,m}(\{\mathsf{H}_{r,k}\}^c). \end{aligned}$$

By (2.22) we see that the right hand side converges to 0 as $k \to \infty$. These prove (2.7). $\square$

Let $\mathcal{H}_r^N = \mathcal{H}_{S_r^N}^{\Phi^N,\Psi^N}$ and $c_9(N) = \int_{\mathsf{S}_r^{N,m}} e^{-\mathcal{H}_r^N(\mathsf{x})}\Lambda(d\mathsf{x})$.

Lemma 4.3. *Suppose $\mu(\mathsf{S}_r^m) > 0$. Then there exists an N_0 such that*

$$\sup_{N_0 \le N \in \mathbb{N}} \max\{c_9^{-1}(N), c_9(N)\} < \infty. \tag{4.2}$$

Proof. By (A.4) and (2.19) we see that $c_{10} := \sup\{-\mathcal{H}_r^N(\mathsf{x}); N \in \mathbb{N}, \mathsf{x} \in \mathsf{S}_r^{N,m}\} < \infty$. Hence $c_9(N) \le e^{c_{10}}$. By (2.16), (2.17), (2.19), and the bounded convergence theorem, we have

$$\lim_{N\to\infty} \int_{\mathsf{S}_r^{N,m}} e^{-\mathcal{H}_r^N(\mathsf{x})}\Lambda(d\mathsf{x}) = \int_{\mathsf{S}_r^m} e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x}) > 0. \tag{4.3}$$

Hence we have $\liminf c_9(N) > 0$. Combining these yields (4.2). $\square$

Let Ψ^N, S_r^N, and S_{rs}^N be as in (A.4) and (A.5), respectively. Let $S_{r\infty}^N = (\cup_{s=r}^{\infty} S_s^N) \backslash S_r^N$. For $\mathsf{x}, \mathsf{s} \in \mathsf{S}$ we write $\mathsf{x} = \sum \delta_{x_i}$ and $\mathsf{s} = \sum \delta_{s_j}$. We set

$$\begin{aligned} \Psi_{r,st}^N(\mathsf{x},\mathsf{s}) &= \sum_{x_i \in S_r^N,\ s_j \in S_{st}^N} \Psi^N(x_i, s_j) && \text{for } r \le s < t \le \infty \\ \Psi_{rs,s\infty}^N(\mathsf{x},\mathsf{s}) &= \sum_{x_i \in S_{rs}^N,\ s_j \in S_{s\infty}^N} \Psi^N(x_i, s_j) && \text{for } r < s < \infty. \end{aligned} \tag{4.4}$$

For a subset A and $m \in \mathbb{N}$ we set $\mathsf{A}^m = \{\mathsf{s} \in \mathsf{S}; \mathsf{s}(A) = m\}$. For $\mathsf{s}, \mathsf{t} \in \mathsf{A}^m$ we set $d_{\mathsf{A}^m}(\mathsf{s},\mathsf{t}) = \min \sum_{i=1}^m |s_i - t_i|$, where the minimum is taken over the labeling such that $\pi_A(\mathsf{s}) = \sum_i \delta_{s_i}$ and $\pi_A(\mathsf{t}) = \sum_i \delta_{t_i}$. The following lemma is crucial in controlling the unintegrability at infinity of logarithmic potentials.

Lemma 4.4. (1) *Let $c_{11} = mk \sup_{N\in\mathbb{N}} \operatorname{diam}(S_r^N)$, where* diam *means the diameter. Then*

$$\sup_{\mathsf{s}\in\mathsf{H}_{r,k}} \sup_{N\in\mathbb{N}} \sup_{r<s\in\mathbb{N}} \sup_{\mathsf{x},\mathsf{x}'\in\mathsf{S}_r^{N,m}} |\Psi_{r,rs}^N(\mathsf{x},\mathsf{s}) - \Psi_{r,rs}^N(\mathsf{x}',\mathsf{s})| \le c_{11} \tag{4.5}$$

$$\sup_{\mathsf{s}\in\mathsf{H}_{r,k}} \sup_{N\in\mathbb{N}} \sup_{r<s\in\mathbb{N}} \sup_{\mathsf{x},\mathsf{x}'\in\mathsf{S}_r^{N,m}} |\Psi_{r,s\infty}^N(\mathsf{x},\mathsf{s}) - \Psi_{r,s\infty}^N(\mathsf{x}',\mathsf{s})| \le c_{11}. \tag{4.6}$$

(2) *Let $r < s \in \mathbb{N}$ and $l \in \mathbb{N}$ be fixed. Let $\mathsf{S}_{rs}^{N,n} = \{\mathsf{x} \in \mathsf{S};\ \mathsf{x}(S_{rs}^N) = n\}$. Then*

$$\sup_{\mathsf{s}\in\mathsf{H}_{s,l}} \sup_{N\in\mathbb{N}} \sup_{\mathsf{y},\mathsf{y}'\in\mathsf{S}_{rs}^{N,n}} |\Psi_{rs,s\infty}^N(\mathsf{y},\mathsf{s}) - \Psi_{rs,s\infty}^N(\mathsf{y}',\mathsf{s})| / d_{\mathsf{S}_{rs}^{N,n}}(\mathsf{y},\mathsf{y}') \le l. \tag{4.7}$$

Proof. We first remark that $\sup_{N\in\mathbb{N}}\mathrm{diam}(S_r^N)<\infty$ by (2.6) and (2.19). We obtain (1) immediately by (2.20), (2.21) and (4.4). Indeed, by (2.20) and (4.4) we see that for all $\mathsf{x}=\sum\delta_{x_i}$ and $\mathsf{x}'=\sum\delta_{x_i'}$ in $\mathsf{S}_r^{N,m}$

$$|\Psi_{r,rs}^N(\mathsf{x},\mathsf{s})-\Psi_{r,rs}^N(\mathsf{x}',\mathsf{s})|\le m\sup_{x,x'\in S_r^N}|\mathsf{h}_{r,s}^N(x,x',\mathsf{s})|.$$

This combined with the definition of $\mathsf{H}_{r,k}$ yields (4.5). The proof of (4.6) is similar. (2) is immediate from (2.21) and the definition of the metric $d_{\mathsf{S}_{rs}^{N,n}}$. □

Let $\mu_{r,k,\mathsf{s},rs}^{N,m}$ denote the conditional probability of $\mu_{r,k}^{N,m}$ given by

$$\mu_{r,k,\mathsf{s},rs}^{N,m}(d\mathsf{x})=\mu_{r,k}^{N,m}(\pi_{S_r^N}\in d\mathsf{x}|\,\pi_{S_{rs}^N}(\mathsf{s})).$$

By construction we have

$$\mu_{r,k}^{N,m}\circ\pi_{S_r^N}^{-1}(d\mathsf{x})=\int_{\mathsf{S}}\mu_{r,k,\mathsf{s},rs}^{N,m}(d\mathsf{x})\,\mu_{r,k}^{N,m}\circ\pi_{S_{rs}^N}^{-1}(d\mathsf{s}). \tag{4.8}$$

By (A.4), μ^N is a (Φ^N,Ψ^N)-canonical Gibbs measure. Hence $\mu_{r,k,\mathsf{s},rs}^{N,m}$ is absolutely continuous with respect to $e^{-\mathcal{H}_r^N(\mathsf{x})}\Lambda(d\mathsf{x})$. So denote its density by $\sigma_{r,k,\mathsf{s},rs}^{N,m}$: for $\mu_{r,k}^{N,m}$-a.e. s

$$\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x})e^{-\mathcal{H}_r^N(\mathsf{x})}\Lambda(d\mathsf{x})=\mu_{r,k,\mathsf{s},rs}^{N,m}(d\mathsf{x}). \tag{4.9}$$

Moreover, for $\mu_{r,k}^{N,m}$-a.e. s, the density $\sigma_{r,k,\mathsf{s},rs}^{N,m}$ is expressed such that

$$\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x})=e^{-\Psi_{r,rs}^N(\mathsf{x},\mathsf{s})}\tau_{r,rs}^N(\mathsf{x},\mathsf{s})/c_{12}^N(\mathsf{s}). \tag{4.10}$$

Here $\tau_{r,rs}^N(\mathsf{x},\mathsf{s})$ and $c_{12}^N(\mathsf{s})$ are defined by

$$\tau_{r,rs}^N(\mathsf{x},\mathsf{s})=1_{\mathsf{S}_r^{N,m}}(\mathsf{x})\int_{\mathsf{S}}1_{\mathsf{H}_{r,k}}(\pi_{S_{rs}^N}(\mathsf{s})+\mathsf{z})e^{-\Psi_{r,s\infty}^N(\mathsf{x},\mathsf{z})-\Psi_{rs,s\infty}^N(\mathsf{s},\mathsf{z})}\mu_{r,k}^{N,m}\circ\pi_{S_{s\infty}^N}^{-1}(d\mathsf{z}), \tag{4.11}$$

$$c_{12}^N(\mathsf{s})=\int_{\mathsf{S}}e^{-\Psi_{r,rs}^N(\mathsf{x},\mathsf{s})}\tau_{r,rs}^N(\mathsf{x},\mathsf{s})e^{-\mathcal{H}_r^N(\mathsf{x})}\Lambda(d\mathsf{x}). \tag{4.12}$$

Lemma 4.5. *Let N_0 be as in Lemma 4.3. Then there exists a positive and finite constant $c_{13}=c_{13}(r,m,k)$ such that for $\mu_{r,k}^{N,m}$-a.e.* s

$$c_{13}^{-1}\le\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x})\le c_{13}\quad\textit{for all } \mathsf{x}\in\mathsf{S}_r^{N,m},\ r<s\in\mathbb{N},\ \textit{and } N_0\le N\in\mathbb{N}. \tag{4.13}$$

Proof. By (4.6) and (4.11) we have for $\mu_{r,k}^{N,m}$-a.e. s

$$\sup_{N\in\mathbb{N}}\sup_{r<s\in\mathbb{N}}\sup_{\mathsf{x},\mathsf{x}'\in\mathsf{S}_r^{N,m}}\tau_{r,rs}^N(\mathsf{x},\mathsf{s})/\tau_{r,rs}^N(\mathsf{x}',\mathsf{s})\le e^{c_{11}}. \tag{4.14}$$

By (4.10) we see that

$$\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x})/\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x}')=e^{-\Psi_{r,rs}^N(\mathsf{x},\mathsf{s})+\Psi_{r,rs}^N(\mathsf{x}',\mathsf{s})}\tau_{r,rs}^N(\mathsf{x},\mathsf{s})/\tau_{r,rs}^N(\mathsf{x}',\mathsf{s}). \tag{4.15}$$

Substituting (4.5) and (4.14) into (4.15) implies for $\mu_{r,k}^{N,m}$-a.e. s

$$\sup_{N\in\mathbb{N}}\sup_{r<s\in\mathbb{N}}\sup_{\mathsf{x},\mathsf{x}'\in\mathsf{S}_r^{N,m}}\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x})/\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x}')\le e^{2c_{11}}.$$

Hence for $\mu_{r,k}^{N,m}$-a.e. s we see that for all $\mathsf{x},\mathsf{x}' \in \mathsf{S}_r^{N,m}$, $r < s \in \mathbb{N}$, and $N \in \mathbb{N}$

$$e^{-2c_{11}}\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x}') \le \sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x}) \le e^{2c_{11}}\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x}'). \tag{4.16}$$

Multiply (4.16) by $1_{\mathsf{S}_r^{N,m}}(\mathsf{x}')e^{-\mathcal{H}_r^N(\mathsf{x}')}$ and integrate with respect to $\Lambda(d\mathsf{x}')$. Then by (4.9) we deduce that for $\mu_{r,k}^{N,m}$-a.e. s

$$e^{-2c_{11}} \le \sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x})\int_{\mathsf{S}_r^{N,m}} e^{-\mathcal{H}_r^N(\mathsf{x}')}\Lambda(d\mathsf{x}') \le e^{2c_{11}} \quad \text{for all } \mathsf{x}\in\mathsf{S}_r^{N,m}. \tag{4.17}$$

This combined with (4.2) yields (4.13) with $c_{13} = e^{2c_{11}}\sup_{N_0\le N\in\mathbb{N}}\max\{c_9^{-1}(N), c_9(N)\}$. $\square$

Let $\mathcal{H}_{rs}^N = \mathcal{H}_{S_{rs}^N}^{\Phi^N,\Psi^N}$ and $\mathcal{H}_{rs} = \mathcal{H}_{S_{rs}}^{\Phi,\Psi}$. By (2.14) and (2.15) we see that $\mu_{r,k}^{N,m}\circ\pi_{S_{rs}^N}^{-1}$ and $\mu_{r,k}^m\circ\pi_{S_{rs}}^{-1}$ are absolutely continuous with respect to Λ, respectively. Hence let Δ^N and Δ denote their Radon-Nikodym densities with respect to $e^{-\mathcal{H}_{rs}^N}\Lambda$ and $e^{-\mathcal{H}_{rs}}\Lambda$, respectively.

Lemma 4.6. (1) *$\mu_{r,k}^{N,m}\circ\pi_{S_{rs}^N}^{-1}$ converge weakly to $\mu_{r,k}^m\circ\pi_{S_{rs}}^{-1}$ as $N\to\infty$.*
(2) *$\Delta^N e^{-\mathcal{H}_{rs}^N}$ converge to $\Delta e^{-\mathcal{H}_{rs}}$ in $L^1(\mathsf{S},\Lambda)$ as $N\to\infty$.*

Proof. Let $E = \{\mathsf{s}\in\mathsf{S}\,;\,\lim_{N\to\infty}S_{rs}^N(\mathsf{s}_N)\ne S_{rs}(\mathsf{s})$ for some $\{\mathsf{s}_N\}$ such that $\lim\mathsf{s}_N = \mathsf{s}\}$. Then by (A.1) and (2.19) we see $\mu(E)=0$. It is known that this implies (1) (see p. 79 [2]).

By (1) it is enough for (2) to show that the sequence $\{\Delta^N e^{-\mathcal{H}_{rs}^N}\}_{N\in\mathbb{N}}$ is relatively compact in $L^1(\mathsf{S},\Lambda)$. By (2.14) and (2.15) we see that $c_{14} := \sup_{N\in\mathbb{N}}\sup_{\mathsf{S}}\Delta^N e^{-\mathcal{H}_{rs}^N} < \infty$. Hence this follows from the relative compactness of $\{\Delta^N e^{-\mathcal{H}_{rs}^N}1_{\mathsf{S}_{rs}^{N,n}}\}_{N\in\mathbb{N}}$ in $L^1(\mathsf{S},\Lambda)$ for each $n\in\mathbb{N}$.

Let $\mathsf{H}_{s,l}$ be as in (2.21). We set $\mu_{nl}^N = \mu_{r,k}^{N,m}(\cdot\cap\mathsf{S}_{rs}^{N,n}\cap\mathsf{H}_{s,l})$. Let Δ_{nl}^N denote the Radon-Nikodym density of $\mu_{nl}^N\circ\pi_{S_{rs}^N}^{-1}$ with respect to $e^{-\mathcal{H}_{rs}^N}\Lambda$. Since $\mu_{nl}^N \le \mu_{r,k}^{N,m}$, we see that

$$\Delta_{nl}^N e^{-\mathcal{H}_{rs}^N} \le \Delta^N e^{-\mathcal{H}_{rs}^N}.$$

Combining this with (2.22) yields

$$\lim_{l\to\infty}\limsup_{N\in\mathbb{N}}\|\Delta^N e^{-\mathcal{H}_{rs}^N} - \Delta_{nl}^N e^{-\mathcal{H}_{rs}^N}\|_{L^1(\mathsf{S},\Lambda)} \le \lim_{l\to\infty}\limsup_{N\in\mathbb{N}}\mu_{r,k}^{N,m}(\mathsf{H}_{s,l}^c) = 0. \tag{4.18}$$

Hence it only remains to prove the relative compactness in $L^1(\mathsf{S},\Lambda)$ of $\{\Delta_{nl}^N e^{-\mathcal{H}_{rs}^N}\}_{N\in\mathbb{N}}$ for each $n,l\in\mathbb{N}$. Let $S^{N,q} = S_{rs}^N\cap\{|s - S_r^N| > 1/q\}$ and set

$$\mathsf{S}_{nl}^N(q) = \{s\in\mathsf{S}_{rs}^{N,n}\cap\mathsf{H}_{s,l}\,;\,\mathsf{s}(S_{rs}^N\backslash S^{N,q}) = 0\}.$$

We define $\mathsf{S}_{nl}(q)$ similarly with the replacement of $\mathsf{S}_{rs}^{N,n}$, S_{rs}^N, and S_r^N by S_{rs}^n, S_{rs}, and S_r, respectively. Then since $\{\Delta_{nl}^N e^{-\mathcal{H}_{rs}^N}\}_{N\in\mathbb{N}}$ is bounded by c_{14}, the relative compactness as above follows from that of $\{\Delta_{nl}^N e^{-\mathcal{H}_{rs}^N}1_{\mathsf{S}_{nl}^N(q)}\}_{N\in\mathbb{N}}$ for all sufficiently large $q\in\mathbb{N}$. Let

$$\psi(q) = \sup_{N\in\mathbb{N}}\sup\{|\Psi_{r,rs}^N(\mathsf{x},\mathsf{y}) - \Psi_{r,rs}^N(\mathsf{x},\mathsf{y}')|/d_{\mathsf{S}_{rs}^{N,n}}(\mathsf{y},\mathsf{y}');\mathsf{x}\in\mathsf{S}_r^{N,m},\ \mathsf{y},\mathsf{y}'\in\mathsf{S}_{nl}^N(q)\}. \tag{4.19}$$

Note that

$$\Delta_{nl}^N(\mathsf{y}) = \text{const.}\int_{\mathsf{S}} 1_{\mathsf{H}_{r,k}\cap\mathsf{H}_{s,l}}(\pi_{S_{rs}^N}(\mathsf{y})+\mathsf{z})e^{-\Psi_{r,rs}^N(\mathsf{z},\mathsf{y})-\Psi_{rs,s\infty}^N(\mathsf{y},\mathsf{z})}\mu_{nl}^N\circ\pi_{S_r^N\cup S_{s\infty}^N}^{-1}(d\mathsf{z}).$$

Then we deduce from (4.19) and Lemma 4.4 (2) the following.

$$\sup_{N\in\mathbb{N}}\sup_{\mathsf{y},\mathsf{y}'\in\mathsf{S}_{nl}^N(q)}\Delta_{nl}^N(\mathsf{y})/\Delta_{nl}^N(\mathsf{y}') \le e^{(\psi(q)+l)d_{\mathsf{S}_{rs}^{N,n}}(\mathsf{y},\mathsf{y}')} \quad \text{for all } q\in\mathbb{N}. \tag{4.20}$$

Hence $\{\Delta_{nl}^N\}_{N\in\mathbb{N}}$ is equi-continuous on $\mathsf{S}_{nl}^N(q)$ for each $q\in\mathbb{N}$. This deduces that $\{\Delta_{nl}^N 1_{\mathsf{S}_{nl}^N(q)}\}_{N\in\mathbb{N}}$ is bounded in $L^\infty(\mathsf{S},\Lambda)$ for sufficiently large $q\in\mathbb{N}$ because $\{\Delta_{nl}^N e^{-\mathcal{H}_{rs}^N}1_{\mathsf{S}_{nl}^N(q)}\}_{N\in\mathbb{N}}$ is bounded in $L^1(\mathsf{S},\Lambda)$ and $\{e^{-\mathcal{H}_{rs}^N}1_{\mathsf{S}_{nl}^N(q)}\}_{N\in\mathbb{N}}$ converges in $L^1(\mathsf{S},\Lambda)$ to the limit $e^{-\mathcal{H}_{rs}}1_{\mathsf{S}_{nl}(q)}$ such that $\|e^{-\mathcal{H}_{rs}}1_{\mathsf{S}_{nl}(q)}\|_{L^1(\mathsf{S},\Lambda)}\neq 0$ for large q. Here we used (2.16), (2.17), and (2.19).

These combined with (2.19) and the Ascoli-Arzelá theorem imply that $\{\Delta_{nl}^N e^{-\mathcal{H}_{rs}^N}1_{\mathsf{S}_{nl}^N(q)}\}_{N\in\mathbb{N}}$ is relatively compact in $L^1(\mathsf{S},\Lambda)$. □

Lemma 4.7. *Let $\mu_{r,k,\mathsf{s},rs}^m$ denote the conditional probability of $\mu_{r,k}^m$ given by*

$$\mu_{r,k,\mathsf{s},rs}^m(d\mathsf{x})=\mu_{r,k}^m(\pi_{S_r}(\mathsf{s})\in d\mathsf{x}|\,\pi_{S_{rs}}(\mathsf{s})).$$

Then we have the following.
(1) *$\mu_{r,k,\mathsf{s},rs}^m$ is absolutely continuous with respect to $e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x})$ for $\mu_{r,k}^m$-a.e. s.*
(2) *For each $r,m,k\in\mathbb{N}$, the Radon-Nikodym densities $\sigma_{r,k,\mathsf{s},rs}^m$ of $\mu_{r,k,\mathsf{s},rs}^m$ in (1) satisfy for $\mu_{r,k}^m$-a.e. s and all $s\in\mathbb{N}$ such that $r<s$*

$$c_{13}^{-1}\le\sigma_{r,k,\mathsf{s},rs}^m(\mathsf{x})\le c_{13}\quad \text{for } \mu_{r,k,\mathsf{s},rs}^m\text{-a.e. } \mathsf{x}. \tag{4.21}$$

Proof. Similarly as Lemma 4.6 (1), we see that $\mu_{r,k}^{N,m}\circ(\pi_{S_r^N},\pi_{S_{rs}^N})^{-1}$ converge weakly to $\mu_{r,k}^m\circ(\pi_{S_r},\pi_{S_{rs}})^{-1}$ as $N\to\infty$. Hence for $\mathsf{f},\mathsf{g}\in C_b(\mathsf{S})$ we have

$$\int_{\mathsf{S}}\mathsf{f}(\pi_{S_r}(\mathsf{s}))\mathsf{g}(\pi_{S_{rs}}(\mathsf{s}))d\mu_{r,k}^m=\lim_{N\to\infty}\int_{\mathsf{S}}\mathsf{f}(\pi_{S_r^N}(\mathsf{s}))\mathsf{g}(\pi_{S_{rs}^N}(\mathsf{s}))d\mu_{r,k}^{N,m}. \tag{4.22}$$

By Lemma 4.5 and the diagonal argument, there exist subsequences of $\{\sigma_{r,k,\mathsf{s},rs}^{N,m}\}_N$, denoted by the same symbol, with a limit $\sigma_{r,k,\mathsf{s},rs}^m$ such that for all $k,m,r<s\in\mathbb{N}$

$$\lim_{N\to\infty}\sigma_{r,k,\mathsf{s},rs}^{N,m}(\pi_{S_r^N}(\mathsf{s}))=\sigma_{r,k,\mathsf{s},rs}^m(\pi_{S_r}(\mathsf{s}))\quad *\text{-weakly in } L^\infty(\mathsf{S},\Lambda). \tag{4.23}$$

Here $\sigma_{r,k,\mathsf{s},rs}^m$ is a function such that $\sigma_{r,k,\mathsf{s},rs}^m(\mathsf{x})=\sigma_{r,k,\pi_{S_{rs}}(\mathsf{s}),rs}^m(\pi_{S_r}(\mathsf{x}))$. Let

$$\mathsf{F}^N(\mathsf{s})=\mathsf{f}(\pi_{S_r^N}(\mathsf{s}))\mathsf{g}(\pi_{S_{rs}^N}(\mathsf{s}))\Delta^N(\mathsf{s})e^{-\mathcal{H}_r^N(\mathsf{s})} \tag{4.24}$$

$$\mathsf{F}(\mathsf{s})=\mathsf{f}(\pi_{S_r}(\mathsf{s}))\mathsf{g}(\pi_{S_{rs}}(\mathsf{s}))\Delta(\mathsf{s})e^{-\mathcal{H}_r(\mathsf{s})}. \tag{4.25}$$

Then by (2.19) and Lemma 4.6 (2) we see that F^N converge to F in $L^1(\mathsf{S},\Lambda)$. This combined with (4.23) implies

$$\lim_{N\to\infty}\int_{\mathsf{S}}\mathsf{F}^N(\mathsf{s})\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{s})d\Lambda=\int_{\mathsf{S}}\mathsf{F}(\mathsf{s})\sigma_{r,k,\mathsf{s},rs}^m(\mathsf{s})d\Lambda. \tag{4.26}$$

By (4.22), (4.26) and $\Delta(\mathsf{y})e^{-\mathcal{H}_r(\mathsf{y})}\Lambda(d\mathsf{y})=\mu_{r,k}^m\circ\pi_{S_{rs}}^{-1}(d\mathsf{y})$, we obtain

$$\int_{\mathsf{S}}\mathsf{f}(\mathsf{x})\mathsf{g}(\mathsf{y})d\mu_{r,k}^m=\int_{\mathsf{S}}\mathsf{f}(\mathsf{x})\mathsf{g}(\mathsf{y})\sigma_{r,k,\mathsf{s},rs}^m(\mathsf{x})e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x})\mu_{r,k}^m\circ\pi_{S_{rs}}^{-1}(d\mathsf{y}),$$

where $\mathsf{x}=\pi_{S_r}(\mathsf{s})$ and $\mathsf{y}=\pi_{S_{rs}}(\mathsf{s})$. Hence we obtain (1) with density $\sigma_{r,k,\mathsf{s},rs}^m$.

By (4.13) and (4.23) we see that $\sigma_{r,k,\mathsf{s},rs}^m$ satisfies (4.21), which implies (2). □

Lemma 4.8. *Let $\mu_{r,k,\mathsf{s}}^m(d\mathsf{x})$ be as in (2.9). Let $\sigma_{r,k,\mathsf{s},rs}^m$ be as in Lemma 4.7. Then the limit*

$$\sigma_{r,k,\mathsf{s}}^m(\mathsf{x}):=\lim_{s\to\infty}\sigma_{r,k,\mathsf{s},rs}^m(\mathsf{x})\quad \mu_{r,k,\mathsf{s}}^m\text{-a.s. } \mathsf{x} \tag{4.27}$$

exists for $\mu_{r,k}^m$-a.s. s. Moreover, $\sigma_{r,k,\mathsf{s}}^m$ satisfies $\mu_{r,k}^m$-a.e. s

$$c_{13}^{-1}\le\sigma_{r,k,\mathsf{s}}^m(\mathsf{x})\le c_{13}\quad \text{for } \mu_{r,k,\mathsf{s}}^m\text{-a.e. } \mathsf{x} \tag{4.28}$$

$$\sigma_{r,k,\mathsf{s}}^m(\mathsf{x})e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x})=\mu_{r,k,\mathsf{s}}^m(d\mathsf{x}). \tag{4.29}$$

Proof. Define $M_s : \mathsf{S} \to \mathbb{R}$ by $M_s(\mathsf{s}) = \sigma^m_{r,k,\mathsf{s},rs}(\mathsf{x})$, where $\mathsf{x} = \pi_{S_r}(\mathsf{s})$. Recall that $\sigma^m_{r,k,\mathsf{s},rs}$ is the Radon-Nikodym density of $\mu^m_{r,k,\mathsf{s},rs}$ with respect to $e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x})$. Hence

$$M_s(\mathsf{s})e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x}) = \mu^m_{r,k,\pi_{S_{rs}}(\mathsf{s}),rs}(d\mathsf{x}). \tag{4.30}$$

Let $\mathcal{F}_s = \sigma[\pi_{S_r}, \pi_{S_{rs}}]$, where $r < s \le \infty$. Then by (4.30) we see that $\{M_s\}_{s\in[r,\infty)}$ is an $(\mathcal{F}_s)$-martingale, which implies $M_\infty(\mathsf{s}) := \lim_{s\to\infty} M_s(\mathsf{s})$ exists for $\mu^m_{r,k}$-a.e. s. Since

$$M_s(\mathsf{s}) = \sigma^m_{r,k,\pi_{S_{rs}}(\mathsf{s}),rs}(\pi_{S_r}(\mathsf{s})),$$

we write the limit as $M_\infty(\mathsf{s}) = \sigma^m_{r,k,\pi_{S_{r\infty}}(\mathsf{s})}(\pi_{S_r}(\mathsf{s}))$. We set $\sigma^m_{r,k,\mathsf{s}}(\mathsf{x}) = \sigma^m_{r,k,\pi_{S_{r\infty}}(\mathsf{s})}(\pi_{S_r}(\mathsf{x}))$. By the disintegration (2.10) this deduces (4.27).

We immediately obtain (4.28) from (4.21) and (4.27).

We see that $\{M_s\}_{s\in[r,\infty)}$ is uniformly integrable by (4.21). Hence by (4.27) we see that $M_s(\mathsf{s})$ converge to $M_\infty(\mathsf{s}) = \sigma^m_{r,k,\mathsf{s}}(\mathsf{x})$ strongly in $L^1(\mathsf{S}^m_r, \mu^m_{r,k,\mathsf{s}})$, which combined with (4.30) and the definition $M_s(\mathsf{s}) = \sigma^m_{r,k,\mathsf{s},rs}(\mathsf{x})$ yields (4.29). $\square$

Proof of Theorem 2.2. Let $\{\mu^m_{r,k}\}$ be as in Lemma 4.2. Then by Lemma 4.2 we see that $\{\mu^m_{r,k}\}$ satisfies (2.7). Moreover by Lemma 4.8 we see that $\mu^m_{r,k,\mathsf{s}}$ satisfies (2.8), which completes the proof of Theorem 2.2. $\square$

5 Proof of Theorem 2.3.

In this section we prove Theorem 2.3. We assume (A.6) throughout the section.

For $\mathsf{y} \in \mathsf{S}$ and $x \in S^N_r$ we set $\mathsf{v}^N_{\ell,r,s}(x,\mathsf{y})$ by

$$\mathsf{v}^N_{\ell,r,s}(x,\mathsf{y}) = \sum_{y_i\in S^N_{rs}} \frac{|\varpi_N(x)|^\ell \cos\ell\angle(\varpi_N(x),\varpi_N(y_i))}{|\varpi_N(y_i)|^\ell} \text{ if } \varpi_N(x) \neq 0, \tag{5.1}$$

and $\mathsf{v}^N_{\ell,r,s} = 0$ if $\varpi_N(x) = 0$. Here $\mathsf{y} = \sum_i \delta_{y_i}$ and $\angle(x,y)$ denotes the angle between the non-zero vectors x and y in $\mathbb{R}^{2d}$. Let

$$\mathsf{V}_{\ell,r,k} = \{\mathsf{y} \in \mathsf{S}\,;\, \sup_{N\in\mathbb{N}} \sup_{r<s\in\mathbb{N}} \sup_{x\neq x'\in S^N_r} \frac{|\mathsf{v}^N_{\ell,r,s}(x,\mathsf{y}) - \mathsf{v}^N_{\ell,r,s}(x',\mathsf{y})|}{|x-x'|} \le k\}. \tag{5.2}$$

Lemma 5.1. *Assume* (2.33) *and that* $\ell_0 \ge 2$. *Then the following holds.*

$$\lim_{k\to\infty} \liminf_{N\to\infty} \mu^N(\mathsf{V}_{\ell,r,k}) = 1 \quad \textit{for all } r \in \mathbb{N},\ 1 \le \ell < \ell_0. \tag{5.3}$$

Proof. Let $(\theta_1(x),\dots,\theta_d(x))$ be the angle of $x \in \mathbb{R}^{2d}\backslash\{0\}$ defined before (2.26). Let $\mathsf{t}_{\mathbf{i},\ell}$ be as in (2.26). Then it is easy to see that for $x, y \neq 0$

$$\begin{aligned} \cos\ell\angle(x,y) &= \prod_{m=1}^d \cos\ell(\theta_m(x)-\theta_m(y)) \\ &= \prod_{m=1}^d \{\cos\ell\theta_m(x)\cdot\cos\ell\theta_m(y) - \sin\ell\theta_m(x)\cdot\sin\ell\theta_m(y)\} \\ &= \sum_{\mathbf{i}\in\mathbb{I}}\{\prod_{m=1}^d i_m\}\,\mathsf{t}_{\mathbf{i},\ell}(x)\,\mathsf{t}_{\mathbf{i},\ell}(y) \qquad \text{(by (2.26)).} \end{aligned} \tag{5.4}$$

Hence by (2.28), (5.1) and (5.4)

$$\mathsf{v}^N_{\ell,r,s}(x,\mathsf{y}) - \mathsf{v}^N_{\ell,r,s}(x',\mathsf{y}) \tag{5.5}$$
$$= \sum_{y_i\in S^N_{rs}} \sum_{\mathbf{i}\in\mathbb{I}} (\prod_{m=1}^d i_m)\ \big[|\varpi_N(x)|^\ell \mathsf{t}_{\mathbf{i},\ell}(\varpi_N(x)) - |\varpi_N(x')|^\ell \mathsf{t}_{\mathbf{i},\ell}(\varpi_N(x'))\big]\ \frac{\mathsf{t}_{\mathbf{i},\ell}(\varpi_N(y_i))}{|\varpi_N(y_i)|^\ell}$$
$$= \sum_{\mathbf{i}\in\mathbb{I}} (\prod_{m=1}^d i_m)\ \big[|\varpi_N(x)|^\ell \mathsf{t}_{\mathbf{i},\ell}(\varpi_N(x)) - |\varpi_N(x')|^\ell \mathsf{t}_{\mathbf{i},\ell}(\varpi_N(x'))\big]\ \mathsf{u}^N_{\ell,\mathbf{i},r,s}(\mathsf{y}).$$

This yields

$$\frac{|\mathsf{v}^N_{\ell,r,s}(x,\mathsf{y}) - \mathsf{v}^N_{\ell,r,s}(x',\mathsf{y})|}{|x-x'|} \le c_{15} \sum_{\mathbf{i}\in\mathbb{I}} |\mathsf{u}^N_{\ell,\mathbf{i},r,s}(\mathsf{y})|. \tag{5.6}$$

Here c_{15} is the constant defined by

$$c_{15} = \sup_{N\in\mathbb{N}} \sup_{x\neq x'\in S^N_r\setminus\{0\}} \frac{\big||\varpi_N(x)|^\ell \mathsf{t}_{\mathbf{i},\ell}(\varpi_N(x)) - |\varpi_N(x')|^\ell \mathsf{t}_{\mathbf{i},\ell}(\varpi_N(x'))\big|}{|x-x'|}.$$

By (2.24) and (2.26) we see that $c_{15} < \infty$. From this, (5.6) and (2.33), we obtain (5.3). □

Lemma 5.2. *Let $x, y \in \mathbb{R}^{2d}$ such that $|x|/|y| < 1$. Let $r = |x|/|y|$. Then*

$$\left|\log|\frac{x}{|y|} - \frac{y}{|y|}|^2 + 2\sum_{\ell=1}^{\ell_0-1} \frac{1}{\ell} r^\ell \cos \ell\angle(x,y)\right| \le \frac{2}{\ell_0} r^{\ell_0}/|1-r|^{\ell_0}. \tag{5.7}$$

Proof. Let $\theta = \angle(x,y)$. We see that

$$\log|\frac{x}{|y|} - \frac{y}{|y|}|^2 = \log\big(1 + r^2 - 2r\cos\theta\big) = \log(1 - re^{\sqrt{-1}\theta}) + \log(1 - re^{-\sqrt{-1}\theta}).$$

Then (5.7) follows from the Tayler expansion. □

Let $v^N_{\ell_0}(x,y) = |\varpi_N(x)|^{\ell_0}/|1 - \frac{|\varpi_N(x)|}{|\varpi_N(y)|}|^{\ell_0}$. Set

$$\bar{\mathsf{v}}^N_{\ell_0,r}(x,x',\mathsf{y}) = \sup_{y_i\in S^N_{r\infty}} |v^N_{\ell_0}(x,y_i) - v^N_{\ell_0}(x',y_i)|, \quad \text{where } \mathsf{y} = \sum_i \delta_{y_i}, \tag{5.8}$$

$$\bar{\mathsf{V}}_{\ell_0,r,k} = \{\mathsf{y}\,;\, \sup_{N\in\mathbb{N}} \sup_{x\neq x'\in S^N_r} |\bar{\mathsf{v}}^N_{\ell_0,r}(x,x',\mathsf{y})|/|x-x'| \le k\}. \tag{5.9}$$

Lemma 5.3. *The following holds.*

$$\lim_{k\to\infty} \liminf_{N\to\infty} \mu^N(\bar{\mathsf{V}}_{\ell_0,r,k}) = 1. \tag{5.10}$$

Proof. Let $f(s,t) = s^{\ell_0}/(1-s/t)^{\ell_0}$ for $0 < s \le b_r \le t$. Let c_{16} be a constant such that

$$\sup_{0<s\le b_r} |\frac{\partial f}{\partial s}(s,t)| \le c_{16}|t-b_r|^{-(\ell_0+1)} \quad \text{for all } b_r \le t < \infty. \tag{5.11}$$

Let $c_{17} = \sup_{N\in\mathbb{N}} \sup_{x\in S^N_r} |\varpi'_N(x)|$. Note that $v^N_{\ell_0}(x,y) = f(\varpi_N(x), \varpi_N(y))$ by definition. Let $c_{18} = (c_{16}c_{17})^{1/2(\ell_0+1)}$. Then we have

$$[\frac{\bar{\mathsf{v}}^N_{\ell_0,r}(x,x',\mathsf{y})}{|x-x'|}]^{1/2(\ell_0+1)} = [\sup_{y_i\in S^N_{r\infty}} \frac{|v^N_{\ell_0}(x,y_i) - v^N_{\ell_0}(x',y_i)|}{|x-x'|}]^{1/2(\ell_0+1)} \tag{5.12}$$
$$\le c_{18} \sup_{y_i\in S^N_{r\infty}} |\varpi_N(y_i) - b_r|^{-1/2} \qquad \text{(by (5.11))}$$
$$\le c_{18}(\{\sum_{y_i\in S^N_{r(r+1)}} |\varpi_N(y_i) - b_r|^{-1/2}\} + |b_{r+1} - b_r|^{-1/2}).$$

This yields

$$E^{\mu^N}[[\frac{\bar{\mathsf{v}}^N_{\ell_0,r}(x,x',\mathsf{y})}{|x-x'|}]^{1/2(\ell_0+1)}] \le c_{18}\{\int_{S^N_{r\,(r+1)}} \frac{1}{|\varpi_N(y)-b_r|^{1/2}}\rho^1_N(y)dy + \frac{1}{|b_{r+1}-b_r|^{1/2}}\}.$$

Hence by (2.15) and (2.24) we have $\sup_{N\in\mathbb{N}} E^{\mu^N}[[\bar{\mathsf{v}}^N_{\ell_0,r}(x,x',\mathsf{y})/|x-x'|]^{1/2(\ell_0+1)}] < \infty$. We thus obtain Lemma 5.3 by Chebychev's inequality. $\square$

We are now in a position to prove Theorem 2.3.

Proof of Theorem 2.3. (2.18) is clear from (2.27). (2.19) follows from (2.24), $\overline{S^{\mathrm{int}}} = S$, and $\Phi < \infty$ a.e.. So it only remains to prove (2.22). We note that (5.3) and (2.32) hold by Lemma 5.1 and (A.7).

Let $\mathsf{h}^N_{r,s}$ be as in (2.20). If $\varpi_N(x) = 0$, then $\mathsf{h}^N_{r,s}(x,\mathsf{y}) = 0$. So we suppose $\varpi_N(x) \neq 0$. By (A.6) and (2.20), we see that

$$\begin{aligned}\mathsf{h}^N_{r,s}(x,\mathsf{y}) &= -\beta \sum_{y_i\in S^N_{rs}} \{\log|\varpi_N(x)-\varpi_N(y_i)| - \log|\varpi_N(y_i)|\} \\ &= -\frac{\beta}{2}\sum_{y_i\in S^N_{rs}} \log|\frac{\varpi_N(x)}{|\varpi_N(y_i)|} - \frac{\varpi_N(y_i)}{|\varpi_N(y_i)|}|^2.\end{aligned}$$

Hence by (5.1) and Lemma 5.2 we have

$$\begin{aligned}|\mathsf{h}^N_{r,s}(x,\mathsf{y}) - \mathsf{h}^N_{r,s}(x',\mathsf{y})| \le \beta\Bigg[&\sum_{\ell=1}^{\ell_0-1}\frac{1}{\ell}\left|\mathsf{v}^N_{\ell,r,s}(x,\mathsf{y}) - \mathsf{v}^N_{\ell,r,s}(x',\mathsf{y})\right| \\ &+\frac{1}{\ell_0}\cdot\sum_{y_i\in S^N_{rs}}|v^N_{\ell_0}(x,y_i) - v^N_{\ell_0}(x',y_i)||\varpi_N(y_i)|^{-\ell_0}\Bigg].\end{aligned} \tag{5.13}$$

Let $A = \{(N,s,x,x'); N\in\mathbb{N}, r<s\in\mathbb{N}, x\neq x'\in S^N_r\}$. By taking the supremum over A, by noting $1\le b_r < \varpi_N(y)$ for $y \in S^N_{r\infty}$ and by using the notations given by (5.1), (5.8) and (2.30), we deduce the following from (5.13).

$$\begin{aligned}\sup_A \frac{|\mathsf{h}^N_{r,s}(x,\mathsf{y}) - \mathsf{h}^N_{r,s}(x',\mathsf{y})|}{|x-x'|} \le \beta\Bigg[&\sum_{\ell=1}^{\ell_0-1}\frac{1}{\ell}\sup_A\frac{|\mathsf{v}^N_{\ell,r,s}(x,\mathsf{y}) - \mathsf{v}^N_{\ell,r,s}(x',\mathsf{y})|}{|x-x'|} \\ &+\frac{1}{\ell_0}\sup_A\frac{|\bar{\mathsf{v}}^N_{\ell_0,r}(x,x',\mathsf{y})|}{|x-x'|}\cdot\sup_{N\in\mathbb{N}}\bar{\mathsf{u}}^N_{\ell_0}(\mathsf{y})\Bigg].\end{aligned} \tag{5.14}$$

Combining (5.14) with (2.21), (5.2) and (2.31), and taking $c_{19} = 1/(\beta\ell_0)$, we obtain

$$\mathsf{H}_{r,k} \supset \left\{\bigcap_{\ell=1}^{\ell_0-1}\mathsf{V}_{\ell,r,c_{19}k}\right\}\bigcap\bar{\mathsf{V}}_{\ell_0,r,\sqrt{k}}\bigcap\bar{\mathsf{U}}_{\ell_0,c_{19}\sqrt{k}}.$$

This together with Lemma 5.1, Lemma 5.3 and (2.32) implies (2.22) in (A.5). $\square$

6 Proof of Theorem 2.4.

In this section we prove Theorem 2.4.

Lemma 6.1. *Let* $\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},r}$ *be as in* (2.34) *with* $j=0$. *Assume* (2.36) *holds for some* $j \ge 1$. *Then* $\{\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},r}\}$ *converges in* $L^1(\mathsf{S},\mu^N)$ *and the limit* $\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},\infty} := \lim_{r\to\infty} \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},r}$ *satisfies*

$$\lim_{r\to\infty} \sup_N \| \sup_{N\in\mathbb{N}} |\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},\infty} - \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},r}| \|_{L^1(\mathsf{S},\mu^N)} = 0. \tag{6.1}$$

Proof. It follows from (2.34) that $\tilde{\mathsf{u}}^{N,j}_{\ell,\mathbf{i},1} = 0$ and $\tilde{\mathsf{u}}^{N,j}_{\ell,\mathbf{i},r} = \sum_{q=2}^r q(\tilde{\mathsf{u}}^{N,j-1}_{\ell,\mathbf{i},q} - \tilde{\mathsf{u}}^{N,j-1}_{\ell,\mathbf{i},q-1})$ for $r\ge 2$. Then by a straightforward calculation we have

$$\tilde{\mathsf{u}}^{N,j-1}_{\ell,\mathbf{i},r} = \frac{\tilde{\mathsf{u}}^{N,j}_{\ell,\mathbf{i},r}}{r} + \sum_{p=2}^{r-1} \frac{\tilde{\mathsf{u}}^{N,j}_{\ell,\mathbf{i},p}}{p(p+1)}. \tag{6.2}$$

Let $\tilde{\mathsf{u}}^{j}_{\ell,\mathbf{i},r}$ be as in (2.35). Then by (6.2) we have

$$\tilde{\mathsf{u}}^{j-1}_{\ell,\mathbf{i},r} \le \frac{\tilde{\mathsf{u}}^{j}_{\ell,\mathbf{i},r}}{r} + \sum_{p=2}^{r-1} \frac{\tilde{\mathsf{u}}^{j}_{\ell,\mathbf{i},p}}{p(p+1)}. \tag{6.3}$$

We now suppose (2.36) holds for j and $c_7 = c_7(j) > 0$. Then by (6.3) we have (2.36) for $j-1$ with a positive constant $c_7(j-1)$. Repeating this we obtain (2.36) for $j=1$ with a positive constant $c_7(1)$. Combining the last result with (6.3) with $j=1$ and then using the Schwartz inequality in the summation yields (6.1). □

Proof of Theorem 2.4. By (6.1) we can and do choose $\{b_r\}$ and $c_{20}>0$ in such a way that

$$\sup_N \| \sup_{N\in\mathbb{N}} |\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},\infty} - \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},b_r}| \|_{L^1(\mathsf{S},\mu^N)} \le c_{20}3^{-r} \quad \text{for all } r\in\mathbb{N}. \tag{6.4}$$

By (2.27) and (2.28) we have $\mathsf{u}^N_{\ell,\mathbf{i},r,s} = \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},b_s} - \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},b_r}$. Then by (2.29) we see that

$$\begin{aligned} \mu^N(\{\mathsf{U}_{\ell,\mathbf{i},r,k}\}^c) &= \mu^N(\{\sup_{N\in\mathbb{N}} \sup_{r<s\in\mathbb{N}} |\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},\infty} - \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},b_r} - (\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},\infty} - \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},b_s})| > k\}) \\ &\le \mu^N(\sup_{N\in\mathbb{N}} |\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},\infty} - \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},b_r}| > k/2) + \sum_{s=r}^{\infty} \mu^N(\sup_{N\in\mathbb{N}} |\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},\infty} - \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},b_s}| > k/2) \\ &\le 2k^{-1}\{\| \sup_{N\in\mathbb{N}} |\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},\infty} - \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},b_r}| \|_{L^1(\mathsf{S},\mu^N)} + \sum_{s=r}^{\infty} \| \sup_{N\in\mathbb{N}} |\tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},\infty} - \tilde{\mathsf{u}}^{N,0}_{\ell,\mathbf{i},b_s}| \|_{L^1(\mathsf{S},\mu^N)}\}. \end{aligned} \tag{6.5}$$

Here we used Chebychev's inequality in the last line. By (6.4) and (6.5) we have

$$\sup_N \mu^N(\{\mathsf{U}_{\ell,\mathbf{i},r,k}\}^c) \le 2k^{-1}c_{20}\{3^{-r} + \frac{3^{-r}}{1-3^{-1}}\}.$$

This immediately implies (2.33). □

7 Translation invariant periodic measures.

In this section we make preparations for a proof of Theorem 2.5.

Let $S=\mathbb{R}^d$. Let $\tau_x : \mathsf{S}\to\mathsf{S}$ be the translation defined by $\tau_x(\mathsf{s}) = \sum_i \delta_{x+s_i}$ for $\mathsf{s}=\sum_i \delta_{s_i}$. We say that a measure ν on S is translation invariant if $\nu\circ\tau_x^{-1} = \nu$ for all $x\in\mathbb{R}^d$. We say that ν is L-periodic if $\nu(\tau_{L\mathbf{e}_i}(\mathsf{s}) = \mathsf{s}) = 1$ for all $i=1,\dots,d$. Moreover, we say that ν is concentrated on A if $\nu(\mathsf{s}(A^c)>0) = 0$. A measure ν on concentrated on $(-L/2, L/2]^d$ can be extended naturally to the L-periodic measure $\bar\nu$ on the configuration space on $\mathbb{R}^d$. We refer to this measure $\bar\nu$ as the L-periodic extension of ν.

Let $\mathbb{T}_N = (-n_N/2, n_N/2]^d$ as before. Throughout this section we assume that ν is concentrated on $\mathbb{T}_N$ and that ν has the n_N-periodic and translation invariant extension. Let ρ_N^n be the n-correlation function of ν. Then, by assumption, $\rho_N^n(x) = 0$ for $x \notin (\mathbb{T}_N)^{n_N}$. Let $\mathcal{T}_N$ be the two level cluster function of ν:

$$\mathcal{T}_N(x,y) = \rho_N^2(x,y) - \rho_N^1(x)\rho_N^1(y). \tag{7.1}$$

Then $\mathcal{T}_N(x,y) = 0$ if $(x,y) \notin (\mathbb{T}_N)^2$. If $(x,y) \in (\mathbb{T}_N)^2$, $\mathcal{T}_N(x,y)$ depends only on $x-y$ modulo $N\mathbf{e}_i$ $(i = 1,\dots,d)$, where $\mathbf{e}_i$ is the ith unit vector. So let $\mathcal{T}_N : \mathbb{R}^d \to \mathbb{R}$ be the n_N-periodic function such that $\mathcal{T}_N(x) = \mathcal{T}_N(x,0)$ for $x \in \mathbb{T}_N$. We set

$$\mathfrak{m}_N(\xi) = \rho_N^1(0) - \mathcal{F}_N(\mathcal{T}_N)(\xi). \tag{7.2}$$

Here $\mathcal{F}_N(f)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi\sqrt{-1}\xi\cdot x} f 1_{\mathbb{T}_N}(x)dx$ denotes the Fourier transform of $f1_{\mathbb{T}_N}$.

Lemma 7.1. *Assume that ν is concentrated on $\mathbb{T}_N$ and that ν has the n_N-periodic and translation invariant extension. Let $h:\mathbb{R}^d \to \mathbb{R}$ be real valued. Set $\mathsf{h}_N(\mathsf{s}) = \sum_{s_i\in\mathbb{T}_N} h(s_i)$, where $\mathsf{s} = \sum_i \delta_{s_i}$. Then*

$$\|\mathsf{h}_N\|^2_{L^2(\mathsf{S},\nu)} = (\rho_N^1(0)\int_{\mathbb{T}_N} h(x)dx)^2 + \frac{1}{(n_N)^d}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}^d/n_N)} |\mathcal{F}_N(h)|^2(\xi)\mathfrak{m}_N(\xi). \tag{7.3}$$

Proof. By $\rho_N^1(x) = \rho_N^1(0)1_{\mathbb{T}_N}(x)$ we see that

$$\int_{\mathsf{S}} \mathsf{h}_N d\nu = \int_{\mathbb{T}_N} h(x)\rho_N^1(x)dx = \rho_N^1(0)\int_{\mathbb{T}_N} h(x)dx. \tag{7.4}$$

Let $\mathrm{Var}^\nu[\mathsf{h}_N]$ be the variance of h_N with respect to ν. By (7.1) and the general property of correlation functions we see that

$$\begin{aligned}\mathrm{Var}^\nu[\mathsf{h}_N] &= \int_{\mathbb{R}^d} h^2(x)\rho_N^1(x)dx - \int_{\mathbb{R}^d\times\mathbb{R}^d} h(x)h(y)\mathcal{T}_N(x,y)dxdy\\ &= \rho_N^1(0)\int_{\mathbb{T}_N} h^2(x)dx - \int_{\mathbb{T}_N\times\mathbb{T}_N} h(x)h(y)\mathcal{T}_N(x-y)dxdy.\end{aligned}$$

We used $\rho_N^1(x) = \rho_N^1(0)1_{\mathbb{T}_N}(x)$ and $\mathcal{T}_N(x,y) = 1_{\mathbb{T}_N}(x)1_{\mathbb{T}_N}(y)\mathcal{T}_N(x-y)$ in the second line. By a direct calculation of the Fourier series we see that

$$\int_{\mathbb{T}_N} h^2(x)dx = \frac{1}{(n_N)^d}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}^d/n_N)} |\mathcal{F}_N(h)(\xi)|^2$$

and

$$\begin{aligned}\int_{\mathbb{T}_N\times\mathbb{T}_N} h(x)h(y)\mathcal{T}_N(x-y)dxdy &= \frac{1}{(n_N)^d}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}^d/n_N)} \mathcal{F}_N(h)(\xi)\overline{\mathcal{F}_N(h*\mathcal{T}_N)(\xi)}\\ &= \frac{1}{(n_N)^d}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}^d/n_N)} |\mathcal{F}_N(h)(\xi)|^2\overline{\mathcal{F}_N(\mathcal{T}_N)(\xi)}\\ &= \frac{1}{(n_N)^d}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}^d/n_N)} |\mathcal{F}_N(h)(\xi)|^2\mathcal{F}_N(\mathcal{T}_N)(\xi).\end{aligned}$$

Here we used the fact that $\mathcal{F}_N(\mathcal{T}_N)$ is real valued because $\mathcal{T}_N(x) = \mathcal{T}_N(-x)$. Combining these with (7.2) yields

$$\mathrm{Var}^\nu[\mathsf{h}_N] = \frac{1}{(n_N)^d}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}^d/n_N)} |\mathcal{F}_N(h)|^2(\xi)\mathfrak{m}_N(\xi). \tag{7.5}$$

(7.3) follows from (7.4) and (7.5) immediately. ∎

8 Proof of Theorems 2.5.

In this section we prove Theorem 2.5 by using the previous results. We begin by defining $\mathsf{K}_{\sin,\beta}$ for $\beta = 1, 4$. For this purpose we recall the standard quaternion notation for 2×2 matrices (see [13, Ch. 2.4]),

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}. \tag{8.1}$$

A quaternion q is represented by $q = q^{(0)}\mathbf{1} + q^{(1)}\mathbf{e}_1 + q^{(2)}\mathbf{e}_2 + q^{(3)}\mathbf{e}_3$. Here the $q^{(i)}$ are complex numbers. There is an identification between the 2×2 complex matrices and the quaternions given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}(a+d)\mathbf{1} - \frac{\sqrt{-1}}{2}(a-d)\mathbf{e}_1 + \frac{1}{2}(b-c)\mathbf{e}_2 - \frac{\sqrt{-1}}{2}(b+c)\mathbf{e}_3. \tag{8.2}$$

We denote by $\Theta(\begin{bmatrix} a & b \\ c & d \end{bmatrix})$ the quaternion defined by the right hand side of (8.2).

For a quaternion $q = q^{(0)}\mathbf{1} + q^{(1)}\mathbf{e}_1 + q^{(2)}\mathbf{e}_2 + q^{(3)}\mathbf{e}_3$, we call $q^{(0)}$ the scalar part of q. A quaternion is called scalar if $q^{(i)} = 0$ for $i = 1, 2, 3$. We often identify a scalar quaternion $q = q^{(0)}\mathbf{1}$ with the complex number $q^{(0)}$ by the obvious correspondence.

Let $\bar{q} = q^{(0)}\mathbf{1} - \{q^{(1)}\mathbf{e}_1 + q^{(2)}\mathbf{e}_2 + q^{(3)}\mathbf{e}_3\}$. A quaternion matrix $A = [a_{ij}]$ is called self-dual if $a_{ij} = \bar{a}_{ji}$ for all i, j. For a self-dual $n\times n$ quaternion matrix $A = [a_{ij}]$ we set

$$\det A = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sign}[\sigma] \prod_{i=1}^{L(\sigma)} [a_{\sigma_i(1)\sigma_i(2)} a_{\sigma_i(2)\sigma_i(3)} \cdots a_{\sigma_i(\ell-1)\sigma_i(\ell)}]^{(0)}. \tag{8.3}$$

Here $\sigma = \sigma_1 \cdots \sigma_{L(\sigma)}$ is a decomposition of σ to products of the cyclic permutations $\{\sigma_i\}$ with disjoint indices. We write $\sigma_i = (\sigma_i(1), \sigma_i(2), \dots, \sigma_i(\ell))$, where ℓ is the length of the cyclic permutation σ_i. The decomposition is unique up to the order of $\{\sigma_i\}$. As before $[\cdot]^{(0)}$ means the scalar part of the quaternion $\cdot$. It is known that the right hand side is well defined. See Section 5.1 in [13] for the details.

We are now ready to introduce $\mathsf{K}_{\sin,\beta}$. Let $S(x) = \sin(\pi x)/\pi x$, $D(x) = \frac{dS}{dx}(x)$ and $I(x) = \int_0^x S(y)dx$. Moreover, let

$$\mathsf{K}_{\sin,1}(x) = \Theta(\begin{bmatrix} S(x) & D(x) \\ I(x) - \frac{1}{2}\mathrm{sgn}(x) & S(x) \end{bmatrix}) \tag{8.4}$$

$$\mathsf{K}_{\sin,2}(x) = S(x) \tag{8.5}$$

$$\mathsf{K}_{\sin,4}(x) = \Theta(\frac{1}{2}\begin{bmatrix} S(2x) & D(2x) \\ I(2x) & S(2x) \end{bmatrix}). \tag{8.6}$$

We thus clarify the meaning of (2.37). In the proof of Theorem 2.5 below we see that the $\mu_{\mathrm{dys},\beta}$ ($\beta = 1, 2, 4$) introduced in Section 2.1 have n-correlation functions $\rho^n_{\mathrm{dys},\beta}$ given by (2.37). (see [13, Ch. (11.3.48), (11.5.16)] [2], [3, Ch. 4, 5] for the details).

Lemma 8.1. *$\mu_{\mathrm{dys},\beta}$ ($\beta = 1, 2, 4$) satisfy* (A.1) *and* (A.3) *with* $(\Phi, \Psi) = (0, -\beta\log|x-y|)$.

[2] $J(r) = I(r) - \frac{1}{2}$ in (11.3.47) in [13] should be $J(r) = I(r) - \frac{1}{2}\mathrm{sgn}(r)$

Proof. Since the correlation functions $\{\rho^n_{\mathrm{dys},\beta}\}$ of $\mu_{\mathrm{dys},\beta}$ have the expression (2.37) and the kernels $\mathsf{K}_{\sin,\beta}$ are bounded, we see that the $\{\rho^n_{\mathrm{dys},\beta}\}$ satisfy (A.1). We see that (A.3) is clear. $\square$

By Lemma 8.1 it only remains to prove (A.2); that is, $\mu_{\mathrm{dys},\beta}$ is a quasi Gibbs measure for $(\Phi,\Psi)=(0,-\beta\log|x-y|)$. For this we recall some facts about circular ensembles.

Let $\check{\nu}^N$ denote the probability measure on $\mathbb{R}^{n_N}$ defined by

$$d\check{\nu}^N = \frac{1}{Z}\prod_{i=1}^{n_N} 1_{\mathbb{T}_N}(x_i) \prod_{i,j=1,i<j}^{n_N} |e^{2\pi\sqrt{-1}x_i/n_N}-e^{2\pi\sqrt{-1}x_j/n_N}|^\beta\, dx_1\cdots dx_{n_N}, \tag{8.7}$$

where Z is the normalization and $\mathbb{T}_N=(-n_N/2,n_N/2]$. It is well known [13], [3] that the distribution of $(e^{2\pi\sqrt{-1}x_i/n_N})_{1\le i\le n_N}$ under $\check{\nu}^N$ is equal to the distribution of the spectrum of the circular orthogonal, unitary and symplectic ensemble for $\beta=1$, 2 and 4, respectively.

Let ι be the map such that $\iota((x_i))=\sum_i \delta_{x_i}$. Set $\nu^N=\check{\nu}^N\circ\iota^{-1}$ and let ϱ^n_N denote the n-correlation function of ν^N. Then by (8.7) we see that $\varrho^n_N=0$ for $n>n_N$ and

$$\varrho^{n_N}_{n_N}(x_1,\ldots,x_{n_N})=\frac{n_N!}{Z}\prod_{i,j=1,i<j}^{n_N} 1_{\mathbb{T}_N}(x_i)|e^{2\pi\sqrt{-1}x_i/n_N}-e^{2\pi\sqrt{-1}x_j/n_N}|^\beta 1_{\mathbb{T}_N}(x_j).$$

For each $n\in\mathbb{N}$, the n-correlation function ϱ^n_N can be written as (see [13, (11.1.10)])

$$\varrho^n_N(x_1,\ldots,x_n)=\det(1_{\mathbb{T}_N}(x_i)\mathsf{K}^N_{\sin,\beta}(x_i-x_j)1_{\mathbb{T}_N}(x_j))_{1\le i,j\le n}, \tag{8.8}$$

where $\mathsf{K}^N_{\sin,\beta}$ is given by (8.4)–(8.6) with the replacement of $S(x)$, $D(x)$, and $I(x)$ by $S_N(x)$, $D_N(x)$, and $I_N(x)$, respectively. Here S_N is defined as

$$S_N(x)=\frac{1}{n_N}\frac{\sin(\pi x)}{\sin(\pi x/n_N)}. \tag{8.9}$$

Moreover, we set $D_N(x)=dS_N(x)/dx$ and $I_N(x)=\int_0^x S_N(y)dy$. One can easily deduce (8.8) and (8.9) from the results in [13, Ch. 11] combined with the scaling $\theta\mapsto 2\pi x/n_N$. Indeed, these follow from (11.1.5), (11.1.6), (11.3.16), (11.3.22), (11.3.23), (11.5.6), and (11.5.13) in [13]. [3]

Lemma 8.2. *Let $n_N=2^{4N}$ and $\mathbb{I}_N=(-N,N)$. Then there exists a smooth injective map $\varpi_N:\mathbb{R}\to\mathbb{R}^2$ satisfying (8.10)–(8.13) below.*

$$\begin{aligned}\varpi_N(x)&=\Big(\frac{n_N}{2\pi}\sin\frac{2\pi x}{n_N},\ \frac{n_N}{2\pi}\{1-\cos\frac{2\pi x}{n_N}\}\Big) && \text{for } x\in\mathbb{I}_N\\ &=(x,0) && \text{for } x\notin\mathbb{I}_{N+1},\end{aligned} \tag{8.10}$$

$$\int_{\mathbb{R}}\sum_{N=1}^\infty|\varpi_N(x)-(x,0)|dx<\infty, \tag{8.11}$$

$$0<\inf_{N\in\mathbb{N}}\inf_{x\in\mathbb{R},x\neq0}|x|/|\varpi_N(x)|,\quad \sup_{N\in\mathbb{N}}\sup_{x\in\mathbb{R},x\neq0}|x|/|\varpi_N(x)|<\infty, \tag{8.12}$$

$$\sup_{r\in\mathbb{N}}\int_{\mathbb{R}}\sum_{N=1}^\infty|1_{\tilde{S}^N_{1r}}(x)-1_{\tilde{S}_{1r}}(x)|dx<\infty. \tag{8.13}$$

Here $\tilde{S}^N_{1r}=\{1\le|\varpi_N(x)|<r\}$ and $\tilde{S}_{1r}=\{1\le|x|<r\}$ as before.

[3] IS_{2N} in (11.1.6) of [13] should be I_{2N}.

Proof. Let ϖ_N be a function defined on $\mathbb{I}_N \cup (\mathbb{I}_{N+1})^c$ satisfying (8.10). Then a direct calculation shows that there exists a constant c_{21} independent of N such that

$$\sup_{x\in\mathbb{I}_N} |\varpi_N(x) - (x,0)| = |\varpi_N(N) - (N,0)| \le c_{21} N^2 2^{-4N} \tag{8.14}$$
$$\sup_{x\in\mathbb{I}_N} |\{\varpi_N(x) - (x,0)\}'| = |\varpi_N'(N) - (1,0)| \le c_{21} N 2^{-4N}.$$

Hence we can extend ϖ_N to $\mathbb{R}$ such that $\varpi_N \in C^\infty(\mathbb{R};\mathbb{R}^2)$ satisfying (8.11), (8.12), and (8.13). $\square$

Lemma 8.3. *Let $\mu^N_{\mathrm{dys},\beta} = \nu^N \circ \pi_{\mathbb{I}_N}^{-1}$. Then $\mu_{\mathrm{dys},\beta}$ satisfy* (A.4) *and* (A.6) *for $\mu^N_{\mathrm{dys},\beta}$. Here $\Phi^N(x) = -\log 1_{\mathbb{I}_N}(x)$ and $\Psi^N(x,y)$ is given by* (2.23) *with ϖ_N in Lemma 8.2.*

Proof. Clearly (2.16), (2.17) and (A.6) are satisfied. By (8.8) and $\mu^N_{\mathrm{dys},\beta} = \nu^N \circ \pi_{\mathbb{I}_N}^{-1}$ the n-correlation function $\rho^n_{N,\mathrm{dys},\beta}$ of $\mu^N_{\mathrm{dys},\beta}$ is given by

$$\rho^n_{N,\mathrm{dys},\beta}(x_1,\ldots,x_n) = \det(1_{\mathbb{I}_N}(x_i)\mathsf{K}^N_{\sin,\beta}(x_i - x_j)1_{\mathbb{I}_N}(x_j))_{1\le i,j\le n}. \tag{8.15}$$

Hence we see that (2.14) and (2.15) are satisfied.

Let $\hat{\Psi}^N(x,y) = -\beta \log|e^{2\pi\sqrt{-1}x/n_N} - e^{2\pi\sqrt{-1}y/n_N}|$. Then by (8.7) we see that ν^N are $(-\log 1_{\mathbb{T}_N}, \hat{\Psi}^N)$-canonical Gibbs measures. We write $\varpi_N = (\varpi_N^1, \varpi_N^2)$. Then by (8.10)

$$\varpi_N^1(x) + \sqrt{-1}\varpi_N^2(x) = \sqrt{-1}\frac{n_N}{2\pi}(1 - e^{2\pi\sqrt{-1}x/n_N}) \quad \text{for } x,y\in\mathbb{I}_N. \tag{8.16}$$

So we have $\hat{\Psi}^N(x,y) = \Psi^N(x,y)$ for $x,y\in\mathbb{I}_N$. Hence $\mu^N_{\mathrm{dys},\beta}$ are (Φ^N,Ψ^N)-canonical Gibbs measures by $\mu^N_{\mathrm{dys},\beta} = \nu^N \circ \pi_{\mathbb{I}_N}^{-1}$. Moreover, $\mu^N_{\mathrm{dys},\beta}(\mathsf{s} \le n_N) = 1$ by construction. We have thus confirmed all the assumptions in (A.4) and (A.6). $\square$

Let $\mathsf{t}_{\mathbf{i},1}$ be as in (2.26). By (2.26) and (8.10) we easily see that for $x,y\in\mathbb{I}_N$

$$\mathsf{t}_{1,1}(\varpi_N(x)) = \cos\frac{\pi x}{n_N}, \quad \mathsf{t}_{-1,1}(\varpi_N(x)) = \sin\frac{\pi x}{n_N}, \quad |\varpi_N(x)| = \frac{n_N}{\pi}\sin\frac{\pi x}{n_N}. \tag{8.17}$$

By $n_N = 2^{4N}$ we see that $\mathbb{I}_N \subset \mathbb{T}_N$ and

$$\sup_{N\in\mathbb{N}}\sup_{x\in\mathbb{I}_N}\frac{|\pi x|}{n_N} \le \frac{\pi}{4}. \tag{8.18}$$

This combined with the third equality in (8.17) yields

$$\mathbb{I}_N \cap \tilde{S}^N_{1r} = \mathbb{I}_N \cap \{\frac{n_N}{\pi}\arcsin\frac{\pi}{n_N} \le |x| < \frac{n_N}{\pi}\arcsin\frac{\pi r}{n_N}\}. \tag{8.19}$$

Lemma 8.4. *Set $\mathbf{i} = (\pm 1)$. Let $u^N_r : \mathbb{R}\to\mathbb{R}$ be such that*

$$u^N_r(x) = \frac{\lceil|\varpi_N(x)|\rceil}{|\varpi_N(x)|}\mathsf{t}_{\mathbf{i},1}(\varpi_N(x))1_{\tilde{S}^N_{1r}}(x). \tag{8.20}$$

Then

$$\sup_{N\in\mathbb{N}}\sup_{r\in\mathbb{N}}\frac{1}{1+\log r}\Big|\int_{\mathbb{T}_N} u^N_r 1_{\mathbb{I}_N}dx\Big| < \infty. \tag{8.21}$$

Proof. We divide the case into two parts. If $\mathbf{i}=(-1)$, then $\int_{\mathbb{T}_N} u_r^N 1_{\mathbb{I}_N} dx = 0$ because $u_r^N 1_{\mathbb{I}_N}$ is an odd function. Hence (8.21) is true. If $\mathbf{i}=(1)$, then by (8.20), (8.17) and $\mathbb{I}_N \subset \mathbb{T}_N$

$$\int_{\mathbb{T}_N} u_r^N 1_{\mathbb{I}_N} dx = \int_{\mathbb{I}_N \cap \tilde{S}_{1r}^N} \frac{\lceil |\varpi_N(x)| \rceil}{|\varpi_N(x)|} \cos \frac{\pi x}{n_N} dx = \int_{\mathbb{I}_N \cap \tilde{S}_{1r}^N} (1 - \frac{1}{|x|} \frac{a_N(x)}{b_N(x)}) \cos \frac{\pi x}{n_N} dx. \tag{8.22}$$

Here $a_N(x) = \lceil |\varpi_N(x)| \rceil - |\varpi_N(x)|$ and $b_N(x) = |\frac{n_N}{\pi x} \sin \frac{\pi x}{n_N}|$. By (8.18)

$$\sup_{N\in\mathbb{N}} \sup_{x\in\mathbb{I}_N \setminus \tilde{S}_1^N} |\frac{a_N(x)}{b_N(x)}| < \infty. \tag{8.23}$$

Combining this with $\tilde{S}_r^N \subset \tilde{S}_{r+1}^N$ for all r, (8.19) and (8.22), we obtain (8.21) for $\mathbf{i}=(1)$. $\square$

Lemma 8.5. *Let u_r^N be as in Lemma 8.4. Let $\mathsf{u}_r^N(\mathsf{s}) = \langle \mathsf{s}, u_r^N \rangle$. Then*

$$\lim_{r\to\infty} r^{-3/4} \sup_{N\in\mathbb{N}} \|\mathsf{u}_r^N\|_{L^2(\mathsf{S}, \mu_{\mathrm{dys},\beta}^N)} = 0. \tag{8.24}$$

Proof. Since $\mu_{\mathrm{dys},\beta}^N = \nu^N \circ \pi_{\mathbb{I}_N}^{-1}$, we have $\|\mathsf{u}_r^N\|_{L^2(\mathsf{S},\mu_{\mathrm{dys},\beta}^N)} = \|\langle \cdot, u_r^N 1_{\mathbb{I}_N}\rangle\|_{L^2(\mathsf{S},\nu^N)}$. Hence (8.24) follows from

$$\lim_{r\to\infty} r^{-3/4} \sup_{N\in\mathbb{N}} \|\langle \cdot, u_r^N 1_{\mathbb{I}_N}\rangle\|_{L^2(\mathsf{S},\nu^N)} = 0. \tag{8.25}$$

Let $P_N = \{-\frac{n_N+1}{2} + p\, ; 1 \le p \le n_N,\ p\in\mathbb{N}\}$. Then by an elementary calculation of the triangle series we have an expansion of $S_N(x)$ such that

$$S_N(x) = \frac{1}{n_N} \sum_{p\in P_N} e^{2\pi x p\sqrt{-1}/n_N}. \tag{8.26}$$

This together with $D_N(x) = dS_N(x)/dx$ and $I_N(x) = \int_0^x S_N(y)dy$ yields

$$D_N(x) = \frac{2\pi\sqrt{-1}}{(n_N)^2} \sum_{p\in P_N} p e^{2\pi x p\sqrt{-1}/n_N} \tag{8.27}$$

$$I_N(x) = \frac{1}{2\pi\sqrt{-1}} \sum_{p\in P_N} \frac{1}{p} \left(e^{2\pi x p\sqrt{-1}/n_N} - 1 \right) = \frac{1}{2\pi\sqrt{-1}} \sum_{p\in P_N} \frac{1}{p}\ e^{2\pi x p\sqrt{-1}/n_N}. \tag{8.28}$$

For (8.28) we use $0 \notin P_N$, which follows from $n_N/2 \in \mathbb{N}$.

Let $\mathcal{T}_\beta^N(x,y)$ be the 2-cluster function of ν^N defined by (7.1). Let $\mathcal{T}_\beta^N(x)$ be the n_N-periodic function such that $\mathcal{T}_\beta^N(x) = \mathcal{T}_\beta^N(x,0)$ for $x\in\mathbb{T}_N$ introduced in Section 7. Then by construction (see (7.1), (8.3), and (8.8)) we have for $x\in\mathbb{T}_N$

$$\mathcal{T}_\beta^N(x) = [\mathsf{K}_{\sin,\beta}^N(x)\mathsf{K}_{\sin,\beta}^N(-x)]^{(0)}. \tag{8.29}$$

Let $P_{N,1} = P_{N,2} = P_N$ and $P_{N,4} = \{p + \frac{1}{2}\, ; p\in\mathbb{N}, -n_N \le N < n_N\}$. Then (8.26)–(8.29) combined with the definition of $\mathsf{K}_{\sin,\beta}^N$ yield

$$\mathcal{T}_2^N(x) = |\mathsf{K}_{\sin,2}^N(x)|^2 = \frac{1}{n_N^2} |\sum_{p\in P_{N,2}} e^{2\pi x p\sqrt{-1}/n_N}|^2 \tag{8.30}$$

$$\mathcal{T}_1^N(x) = \frac{1}{n_N^2} |\sum_{p\in P_{N,1}} e^{2\pi x p\sqrt{-1}/n_N}|^2 - \frac{1}{n_N^2} \sum_{p,q\in P_{N,1}} \frac{p}{q} e^{2\pi x(p+q)\sqrt{-1}/n_N} \tag{8.31}$$

$$\mathcal{T}_4^N(x) = \frac{1}{4n_N^2} |\sum_{p\in P_{N,4}} e^{4\pi x p\sqrt{-1}/n_N}|^2 - \frac{1}{4n_N^2} \sum_{p,q\in P_{N,4}} \frac{p}{q} e^{4\pi x(p+q)\sqrt{-1}/n_N}. \tag{8.32}$$

For reader's convenience, we provide more details of the proof of (8.31) and (8.32) as an Appendix (see Section 10.3).

We now consider the Fourier series $\mathcal{F}_N(\mathcal{T}_\beta^N)(\xi) = \int_{\mathbb{T}_N} e^{-2\pi\sqrt{-1}\xi\cdot x}\mathcal{T}_\beta^N(x)dx$. By (8.30)–(8.32) we obtain $\sup_N \sup_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}/n_N)} |\mathcal{F}_N(\mathcal{T}_\beta^N)(\xi)| < \infty$. Hence $\mathfrak{m}_N$ defined by (7.2) for ν^N satisfies

$$c_{22} := \sup_N \sup_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}/n_N)} |\mathfrak{m}_N(\xi)| < \infty.$$

Since $\sup_{N\in\mathbb{N}} \int_{\mathbb{R}} |u_r^N 1_{\mathbb{I}_N}|^2 dx \le c_{23}r$ for some constant c_{23}, we have

$$\sup_N \frac{1}{n_N} \sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}/n_N)} |\mathcal{F}_N(u_r^N 1_{\mathbb{I}_N})|^2(\xi) \le c_{23}r.$$

Combining these we obtain

$$\sup_N \frac{1}{n_N} \sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}/n_N)} |\mathcal{F}_N(u_r^N 1_{\mathbb{I}_N})|^2(\xi)\mathfrak{m}_N(\xi) \le c_{22}c_{23}r. \tag{8.33}$$

By (8.8) we have $\varrho_N^1(0) = 1$. By this, (8.33), (8.21), and Lemma 7.1, we obtain (8.25). Hence we complete the proof of (8.24). $\square$

Lemma 8.6. *Let u_r^N be as in Lemma 8.5. Let $\tilde{\mathsf{u}}_r(\mathsf{s}) = \sup_{M\in\mathbb{N}} |\mathsf{u}_r^M(\mathsf{s})|$. Then*

$$\lim_{r\to\infty} r^{-3/4} \sup_{N\in\mathbb{N}} \|\tilde{\mathsf{u}}_r\|_{L^1(\mathsf{S},\mu_{\mathrm{dys},\beta}^N)} = 0. \tag{8.34}$$

Proof. We note that $|\tilde{\mathsf{u}}_r| \le \{\sup_{M\in\mathbb{N}} |\mathsf{u}_r^M - \mathsf{u}_r^N|\} + |\mathsf{u}_r^N|$. Then by Lemma 8.5 we deduce (8.34) from

$$\lim_{r\to\infty} r^{-3/4} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{u}_r^M - \mathsf{u}_r^N| \|_{L^1(\mathsf{S},\mu_{\mathrm{dys},\beta}^N)} = 0. \tag{8.35}$$

Let $\varpi(x) = (x, 0)$. We observe that

$$\begin{aligned} u_r^M - u_r^N &= \frac{\lceil|\varpi_M|\rceil}{|\varpi_M|}\mathsf{t}_{\mathbf{i},1}\circ\varpi_M 1_{\tilde{S}_{1r}^M} - \frac{\lceil|\varpi_N|\rceil}{|\varpi_N|}\mathsf{t}_{\mathbf{i},1}\circ\varpi_N 1_{\tilde{S}_{1r}^N} \\ &= \frac{\lceil|\varpi_M|\rceil}{|\varpi_M|}\mathsf{t}_{\mathbf{i},1}\circ\varpi_M(1_{\tilde{S}_{1r}^M} - 1_{\tilde{S}_{1r}}) + \frac{\lceil|\varpi_M|\rceil}{|\varpi_M|}\mathsf{t}_{\mathbf{i},1}\circ\varpi_M(1_{\tilde{S}_{1r}} - 1_{\tilde{S}_{1r}^N}) \\ &\quad + \frac{\lceil|\varpi_M|\rceil}{|\varpi_M|}(\mathsf{t}_{\mathbf{i},1}\circ\varpi_M - \mathsf{t}_{\mathbf{i},1}\circ\varpi)1_{\tilde{S}_{1r}^N} + \frac{\lceil|\varpi_M|\rceil}{|\varpi_M|}(\mathsf{t}_{\mathbf{i},1}\circ\varpi - \mathsf{t}_{\mathbf{i},1}\circ\varpi_N)1_{\tilde{S}_{1r}^N} \\ &\quad + (\frac{\lceil|\varpi_M|\rceil}{|\varpi_M|} - 1)\mathsf{t}_{\mathbf{i},1}\circ\varpi_N 1_{\tilde{S}_{1r}^N} + (1 - \frac{\lceil|\varpi_N|\rceil}{|\varpi_N|})\mathsf{t}_{\mathbf{i},1}\circ\varpi_N 1_{\tilde{S}_{1r}^N} \\ &=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned} \tag{8.36}$$

Hence $|\mathsf{u}_r^M - \mathsf{u}_r^N| = \langle\cdot, J_1\rangle + \cdots + \langle\cdot, J_6\rangle$.

By $|\frac{\lceil|\varpi_M|\rceil}{|\varpi_M|}\mathsf{t}_{\mathbf{i},1}\circ\varpi_M| \le 2$, (8.13), and $\rho_{N,\mathrm{dys},\beta}^1 \le 1$ we have

$$\begin{aligned} \sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\langle\cdot, J_1\rangle| \|_{L^1(\mathsf{S},\mu_{\mathrm{dys},\beta}^N)} &\le \sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}} \|\langle\cdot, \sup_{M\in\mathbb{N}} |J_1|\rangle\|_{L^1(\mathsf{S},\mu_{\mathrm{dys},\beta}^N)} \\ &\le \sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}} \|\langle\cdot, \sum_{M=1}^{\infty} |J_1|\rangle\|_{L^1(\mathsf{S},\mu_{\mathrm{dys},\beta}^N)} \\ &= \sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}} \int_{\mathbb{R}} \sum_{M=1}^{\infty} |J_1|\rho_{N,\mathrm{dys},\beta}^1 dx < \infty. \end{aligned} \tag{8.37}$$

Similarly we have

$$\sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}}\|\sup_{M\in\mathbb{N}}|\langle\cdot,J_2\rangle|\|_{L^1(\mathsf{S},\mu^N_{\mathrm{dys},\beta})}<\infty. \tag{8.38}$$

By (2.26) and (8.11) combined with $|\mathsf{t}'_{\mathbf{i},1}|\le 1$ and $\rho^1_{N,\mathrm{dys},\beta}\le 1$ we have

$$\begin{aligned}
&\sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}}\|\sup_{M\in\mathbb{N}}|\langle\cdot,J_3\rangle|\|_{L^1(\mathsf{S},\mu^N_{\mathrm{dys},\beta})}\le\sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}}\|\langle\cdot,\sup_{M\in\mathbb{N}}|J_3|\rangle\|_{L^1(\mathsf{S},\mu^N_{\mathrm{dys},\beta})}\\
&\le\sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}}\|\langle\cdot,\sum_{M=1}^{\infty}|J_3|\rangle\|_{L^1(\mathsf{S},\mu^N_{\mathrm{dys},\beta})}\\
&=\sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}}\int_{\tilde{S}^N_{1r}}\{\sum_{M=1}^{\infty}|\mathsf{t}_{\mathbf{i},1}\circ\varpi_M-\mathsf{t}_{\mathbf{i},1}\circ\varpi|\}\rho^1_{N,\mathrm{dys},\beta}dx\\
&\le\sup_{N\in\mathbb{N}}\int_{\mathbb{R}}\{\sum_{M=1}^{\infty}|\varpi_M(x)-(x,0)|\}dx<\infty.
\end{aligned} \tag{8.39}$$

Similarly we have

$$\sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}}\|\sup_{M\in\mathbb{N}}|\langle\cdot,J_4\rangle|\|_{L^1(\mathsf{S},\mu^N_{\mathrm{dys},\beta})}<\infty. \tag{8.40}$$

By (8.12) and (8.13) combined with $0\le\lceil|\varpi_M|\rceil-|\varpi_M|<1$ and $\rho^1_{N,\mathrm{dys},\beta}\le 1$ we have

$$\begin{aligned}
&\lim_{r\to\infty}r^{-3/4}\sup_{N\in\mathbb{N}}\|\sup_{M\in\mathbb{N}}|\langle\cdot,J_5\rangle|\|_{L^1(\mathsf{S},\mu^N_{\mathrm{dys},\beta})}\\
&\qquad\le\lim_{r\to\infty}r^{-3/4}\sup_{N\in\mathbb{N}}\int_{\tilde{S}^N_{1r}}\sup_{M\in\mathbb{N}}\frac{1}{|\varpi_M|}\rho^1_{N,\mathrm{dys},\beta}dx=0.
\end{aligned} \tag{8.41}$$

Similarly we have

$$\lim_{r\to\infty}r^{-3/4}\sup_{N\in\mathbb{N}}\|\sup_{M\in\mathbb{N}}|\langle\cdot,J_6\rangle|\|_{L^1(\mathsf{S},\mu^N_{\mathrm{dys},\beta})}=0. \tag{8.42}$$

Putting (8.36)–(8.42) together we obtain (8.35), which completes the proof of (8.34). □

Proof of Theorem 2.5. By Lemma 8.1 we obtain (A.1) and (A.3). In view of Theorem 2.2 it is enough to check (A.4) and (A.5) for (A.2). We have checked (A.4) and (A.6) by Lemma 8.3. Hence by Theorem 2.3 it is sufficient to prove (A.7) for (A.5).

We prove (A.7) is satisfied with $\ell_0=2$ and $b_r=r$. (2.32) and (2.33) in (A.7) are checked below. By (8.8) and (8.9) we see $\rho^1_{N,\mathrm{dys},\beta}(x)=1_{\mathbb{I}_N}(x)$. By (8.12) we see that $c_{24}:=\sup_{N\in\mathbb{N}}\sup_{x\neq 0}|x|/|\varpi_N(x)|<\infty$ and $c_{25}:=\inf_{N\in\mathbb{N}}|\varpi_N^{-1}(1)|>0$. Hence (2.32) follows from

$$\sup_{N\in\mathbb{N}}\int_{\mathsf{S}}\{\sup_{M\in\mathbb{N}}\langle\mathsf{x},\frac{1}{|\varpi_M|^2}1_{\tilde{S}^M_{1,\infty}}\rangle\}\mu^N_{\mathrm{dys},\beta}(d\mathsf{x})\le c_{24}^2\sup_{N\in\mathbb{N}}\int_{c_{25}\le|x|}\frac{1}{|x|^2}dx<\infty.$$

To prove (2.33) we use Theorem 2.4. In view of Theorem 2.4 it only remains to prove (2.36) with $j=\ell=d=1$. This follows from Lemma 8.6, which completes the proof. □

9 Proof of Theorem 2.6.

In this section we prove Theorem 2.6. In the following we often identify $\mathbb{R}^2$ as $\mathbb{C}$ by the correspondence $z=(x,y)\in\mathbb{R}^2\mapsto z=x+\sqrt{-1}y\in\mathbb{C}$. We set $z=|z|e^{\sqrt{-1}\arg z}$.

Lemma 9.1. *$\mu_{\rm gin}$ satisfies assumptions (A.1), (A.3), (A.4) and (A.6). Here we take $\Phi(z) = |z|^2$ and $\Psi(z_1, z_2) = -\beta \log |z_1 - z_2|$.*

Proof. (A.1) is clear from (2.39) and (2.40). (A.3) is clear because $\Phi(z) = |z|^2$ and $\Psi(z) = -2\log |z|$. As for (A.6) we take $\varpi_N(z) = (z, 0)$. Then (A.6) is clearly satisfied.

As for (A.4) we set $n_N = N$, $\Phi^N(z) = |z|^2$ and $\Psi^N(z) = -2 \log |z|$. Then (2.16) and (2.17) are trivial because Φ^N and Ψ^N are independent of N. Let $d\mu^N_{\rm gin} = \check{\mu}^N_{\rm gin} \circ \iota^{-1}$, where $\check{\mu}^N_{\rm gin}$ is the probability measure on $\mathbb{C}^N$ given by

$$\check{\mu}^N_{\rm gin} = \frac{1}{Z} e^{-\sum_{i=1}^N |z_i|^2} \prod_{1\le i<j\le N} |z_i - z_j|^2 dz_1 \cdots dz_N.$$

Then $\mu^N_{\rm gin}$ is a $(|z|^2, -2\log|z|)$-canonical Gibbs measure. Let $\rho^n_{N,{\rm gin}}$ be the n-correlation function of $\mu^N_{\rm gin}$. It is known (cf. p. 943 in [22]) that $\rho^n_{N,{\rm gin}}$ is given by

$$\rho^n_{N,{\rm gin}}(z_1, \dots, z_n) = \det(\mathsf{K}^N_{\rm gin}(z_i, z_j))_{1\le i,j\le n}, \tag{9.1}$$

where $\mathsf{K}^N_{\rm gin}$ is the kernel defined by

$$\mathsf{K}^N_{\rm gin}(z_1, z_2) = \frac{1}{\pi}\{\sum_{k=0}^{N-1} \frac{1}{k!}(z_1 \cdot \bar{z}_2)^k\} \exp\{-\frac{|z_1|^2}{2} - \frac{|z_2|^2}{2}\}. \tag{9.2}$$

By combining these with (2.39) and (2.40) we obtain (2.14) and (2.15). We have thus checked all assumptions of (A.4). □

Next we proceed with the proof of (A.7). For this purpose we first prepare Lemma 9.2. We denote $\langle \mathsf{s}, f\rangle = \sum_i f(s_i)$ for $\mathsf{s} = \sum_i \delta_{s_i}$.

Lemma 9.2. *Let $h_r(z) = 1_{\tilde{S}_r}(z) e^{\sqrt{-1} \arg z}$, where $\tilde{S}_r = \{z \in \mathbb{C}\,; |z| < r\}$ and $\arg z$ is the angle of $z = |z|e^{\sqrt{-1}\arg z} \in \mathbb{C}$. Let $f\colon\mathbb{C}\to\mathbb{C}$ be a bounded, measurable function such that*

$$\sup_{|z|=r} |f(z) - z_0| = O(r^{-1}) \ \text{as } r \to \infty \text{ for some } z_0 \in \mathbb{C}. \tag{9.3}$$

Here $f(r) = O(g(r))$ as $r \to \infty$ means $\limsup_{r\to\infty} |f(r)|/|g(r)| < \infty$. Then we have

$$\sup_N {\rm Var}^{\mu^N_{\rm gin}}(\langle \mathsf{s}, h_r f\rangle) = O(r) \ \text{as } r \to \infty. \tag{9.4}$$

We prove Lemma 9.2 in Section 10.4.

Lemma 9.3. *Let $\mu^N_{\rm gin}$ be as in the proof of Lemma 9.1. Then $\mu^N_{\rm gin}$ $(N \in \mathbb{N})$ satisfy (A.7).*

Proof. We take $b_r = r$ and $\ell_0 = 3$. Recall that assumption (A.7) consists of (2.32) and (2.33). We first check (2.32).

By (9.1) and (9.2) we have $\rho^{N,1}_{\rm gin}(z) \le \rho^1_{\rm gin}(z) = \rho^1_{\rm gin}(0)$. Recall that $\varpi_N(z) = (z, 0)$. Then $\bar{\mathsf{u}}^N_{\ell_0}$ is independent of N. Moreover, $\bar{\mathsf{u}}^N_{\ell_0} = \langle \cdot, |z|^{-3} 1_{[1,\infty)}\rangle$ by (2.30). So we have

$$\int_{\mathsf{S}} (\sup_{M\in\mathbb{N}} \bar{\mathsf{u}}^M_{\ell_0}) d\mu^N_{\rm gin} = \int_{|z|\ge1} \frac{1}{|z|^3} \rho^{N,1}_{\rm gin}(z) dz \le \int_{|z|\ge1} \frac{1}{|z|^3} dz\, \rho^1_{\rm gin}(0) < \infty. \tag{9.5}$$

Hence we obtain (2.32) by Chebychev's inequality.

Let $1 \le j \le \ell < \ell_0 = 3$. Let $\tilde{u}^{N,j}_{\ell,\mathbf{i},r}$ be as in (2.34). We remark that $\tilde{u}^{N,j}_{\ell,\mathbf{i},r}$ is independent of N. We note $\mathbb{I} = \{1,-1\}^2$ because $d = 2$. Since $\varpi_N(z) = (z,0)$, we see that $\tilde{u}^{N,j}_{\ell,(-1,\pm 1),r}(z) = 0$. Moreover, it is easy to see that

$$\tilde{u}^{N,j}_{\ell,(1,1),r}(z) + \sqrt{-1}\tilde{u}^{N,j}_{\ell,(1,-1),r}(z) = \lceil |z| \rceil^j |z|^{-\ell} 1_{\tilde{S}_r \setminus \tilde{S}_1}(z) e^{\sqrt{-1}\arg z}.$$

Take $f(z) = \lceil |z| \rceil^j |z|^{-\ell} 1_{\tilde{S}_r \setminus \tilde{S}_1}(z)$. Then f satisfies (9.3) with $z_0 = 0$ because $0 \le \lceil |z| \rceil \le 1$. Hence by Lemma 9.2 we obtain

$$\lim_{r\to\infty} r^{2c_7 - 2j} \sup_N \mathrm{Var}^{\mu^N_{\mathrm{gin}}}(\tilde{\mathsf{u}}^{N,j}_{\ell,\mathbf{i},r}) = 0 \quad \text{with } c_7 = 1/4, \text{ say.} \tag{9.6}$$

This implies (2.36) because $E^{\mu^N_{\mathrm{gin}}}[\tilde{\mathsf{u}}^{N,j}_{\ell,\mathbf{i},r}] = 0$ and $\tilde{\mathsf{u}}^{N,j}_{\ell,\mathbf{i},r}$ is independent of N. We therefore obtain (2.33) by Theorem 2.4. □

Proof of Theorem 2.6. By Lemma 9.1 it only remains to prove (A.2). For this it is enough to check (A.5) by Theorem 2.2 and Lemma 9.1. In proving (A.5) it is sufficient to prove (A.6) and (A.7) by Theorem 2.3. We obtain (A.6) and (A.7) by Lemma 9.1 and Lemma 9.3, respectively. We have thus proved Theorem 2.6. □

10 Appendix

10.1 Proof of Lemma 3.4.

Proof of Lemma 3.4. Let $\{f_{\mathsf{p}}\}$ be a $\mathcal{E}^{m,a,\mu}_{r,k}$-Cauchy sequence in $\mathcal{D}^{a,\mu}_{\infty}$ such that $\lim \|f_{\mathsf{p}}\|_{L^2(\mathsf{S},\mu^m_{r,k})} = 0$. Then by (3.3) and (3.4) we see that $\{f_{\mathsf{p}}\}$ satisfies

$$\lim_{\mathsf{p},\mathsf{q}\to\infty} \int_{\mathsf{S}} \mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}(f_{\mathsf{p}} - f_{\mathsf{q}}, f_{\mathsf{p}} - f_{\mathsf{q}}) \mu^m_{r,k}(d\mathsf{s}) = 0 \tag{10.1}$$

$$\lim_{\mathsf{p}\to\infty} \int_{\mathsf{S}} \|f_{\mathsf{p}}\|^2_{L^2(\mathsf{S}^m_r, \mu^m_{r,k,\mathsf{s}})} \mu^m_{r,k}(d\mathsf{s}) = 0. \tag{10.2}$$

We prove that $\lim_{\mathsf{p}\to\infty} \mathcal{E}^{m,a,\mu}_{r,k}(f_{\mathsf{p}}, f_{\mathsf{p}}) = 0$. For this purpose it is enough to show that, for any subsequence $\{f_{1,\mathsf{p}}\}$ of $\{f_{\mathsf{p}}\}$, we can choose a subsequence $\{f_{2,\mathsf{p}}\}$ of $\{f_{1,\mathsf{p}}\}$ such that

$$\lim_{\mathsf{p}\to\infty} \mathcal{E}^{m,a,\mu}_{r,k}(f_{2,\mathsf{p}}, f_{2,\mathsf{p}}) = 0. \tag{10.3}$$

So let $\{f_{1,\mathsf{p}}\}$ be any subsequence of $\{f_{\mathsf{p}}\}$. Then by (10.1) and (10.2) we can choose a subsequence $\{f_{2,\mathsf{p}}\}$ such that $\mu^m_{r,k}(\mathsf{A}_{\mathsf{p}}) \le 2^{-k}$ and $\mu^m_{r,k}(\mathsf{B}_{\mathsf{p}}) \le 2^{-k}$, where

$$\mathsf{A}_{\mathsf{p}} = \{\mathsf{s}\,;\, \mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}(f_{2,\mathsf{p}} - f_{2,\mathsf{p}+1}, f_{2,\mathsf{p}} - f_{2,\mathsf{p}+1}) \ge 2^{-2k}\}$$
$$\mathsf{B}_{\mathsf{p}} = \{\mathsf{s}\,;\, \|f_{\mathsf{p}}\|^2_{L^2(\mathsf{S}^m_r, \mu^m_{r,k,\mathsf{s}})} \ge 2^{-2k}\}.$$

Hence by Borel-Cantelli's lemma we see that $\mu^m_{r,k}(\limsup \mathsf{A}_{\mathsf{p}}) = \mu^m_{r,k}(\limsup \mathsf{B}_{\mathsf{p}}) = 0$. This means that, for $\mu^m_{r,k}$-a.s. s, the sequence $\{f_{2,\mathsf{p}}\}$ is an $\mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}$-Cauchy sequence converging to 0 in $L^2(\mathsf{S}^m_r, \mu^m_{r,k,\mathsf{s}})$ as $\mathsf{p} \to \infty$. Therefore by assumption we have

$$\lim_{\mathsf{p}\to\infty} \mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}(f_{2,\mathsf{p}}, f_{2,\mathsf{p}}) = 0 \quad \text{for } \mu^m_{r,k}\text{-a.s. } \mathsf{s}. \tag{10.4}$$

Let $\check{\mu}^m_{r,k,\mathsf{s}}$ be the symmetric measure on S^m_r such that $\check{\mu}^m_{r,k,\mathsf{s}} \circ \iota^{-1} = \mu^m_{r,k,\mathsf{s}}$. For $f_{2,\mathsf{p}}$ there exists a function $f^{r,m}_{2,\mathsf{p}} : S^m_r \times \mathsf{S} \to \mathbb{R}$ such that $f^{r,m}_{2,\mathsf{p}}(\mathbf{x}, \mathsf{s})$ is symmetric in $\mathbf{x} = (x_1, \ldots, x_m)$ for

each $\mathsf{s} \in \mathsf{S}$ and that $f_{2,\mathsf{p}}^{r,m}(\mathbf{x}, \mathsf{s}) = f_{2,\mathsf{p}}(\mathsf{s})$ for $\mathsf{s} \in \mathsf{S}_r^m$ being decomposed as $\mathsf{s} = \iota(\mathbf{x}) + \pi_{S_r^c}(\mathsf{s})$. Let $x_l = (x_{l1}, \dots, x_{ld}) \in \mathbb{R}^d$. Then

$$\int_{\mathsf{S}_r^m} \mathcal{E}_{r,k,\mathsf{s}}^{m,a,\mu}(f_{2,\mathsf{p}} - f_{2,\mathsf{p}+1}, f_{2,\mathsf{p}} - f_{2,\mathsf{p}+1}) \mu_{r,k}^m(d\mathsf{s}) =$$

$$\int_{S_r^m \times \mathsf{S}} \frac{1}{2} \sum_{l=1}^m \sum_{i,j=1}^d a_{ij}(\mathsf{s}, x_l) \frac{\partial (f_{2,\mathsf{p}}^{r,m} - f_{2,\mathsf{p}+1}^{r,m})}{\partial x_{li}} \cdot \frac{\partial (f_{2,\mathsf{p}}^{r,m} - f_{2,\mathsf{p}+1}^{r,m})}{\partial x_{lj}} \check{\mu}_{r,k,\mathsf{s}}^m(d\mathbf{x}) \mu_{r,k}^m(d\mathsf{s}).$$

Hence by (10.1) we see that the vector valued function $(\nabla_{x_l} f_{2,\mathsf{p}}^{r,m})_{l=1,\dots,m} : S_r^m \times \mathsf{S} \to (\mathbb{R}^d)^m$ is a Cauchy sequence in $L^2(S_r^m \times \mathsf{S} \to (\mathbb{R}^d)^m, \check{\mu}_{r,k,\mathsf{s}}^m)$, where we equip $L^2(S_r^m \times \mathsf{S} \to (\mathbb{R}^d)^m, \check{\mu}_{r,k,\mathsf{s}}^m)$ with the inner product

$$(\mathbf{f}, \mathbf{g}) = \int_{S_r^m \times \mathsf{S}} \sum_{l=1}^m \{ f_l(\mathbf{x}, \mathsf{s}) g_l(\mathbf{x}, \mathsf{s}) a_0(\iota(\mathbf{x}) + \pi_{S_r^c}(\mathsf{s}), x_l) \} \check{\mu}_{r,k,\mathsf{s}}^m(d\mathbf{x}) \mu_{r,k}^m(d\mathsf{s}).$$

Here $\mathbf{f} = (f_1, \dots, f_m)$, and a_0 is the function in (2.3). Combining this with (10.1) and (10.4) we obtain (10.3), which completes the proof. □

10.2 Proof of Lemma 4.1

In this subsection we give a proof of Lemma 4.1.

Proof of Lemma 4.1. Let $\tilde{S}_r = \{ s \in S\, ;\, |s| \le r \}$ as before. A permutation invariant function $m_r^n : \tilde{S}_r^n \to \mathbb{R}$ is by definition the n-density function of μ if, for any bounded $\sigma[\pi_{\tilde{S}_r}]$-measurable function f,

$$\int_{\tilde{\mathsf{S}}_r^n} \mathsf{f} d\mu = \frac{1}{n!} \int_{\tilde{S}_r^n} f_r^n m_r^n dx_1 \cdots dx_n,$$

where $\tilde{\mathsf{S}}_r^n = \{ \mathsf{x} \in \mathsf{S}\, ;\, \mathsf{x}(\tilde{S}_r) = n \}$, and $f_r^n : \tilde{S}_r^n \to \mathbb{R}$ is the permutation invariant function such that $f_r^n(x_1, \dots, x_n) = \mathsf{f}(\mathsf{x})$ for $\mathsf{x} \in \tilde{\mathsf{S}}_r^n$ such that $\pi_{\tilde{S}_r}(\mathsf{x}) = \sum_i \delta_{x_i}$.

Let $m_{N,r}^n(x_1, \dots, x_n)$ (resp. $m_r^n(x_1, \dots, x_n)$) be the n-density function of μ^N (resp. μ) on $\tilde{S}_r$. Then by (2.15) we easily see that

$$m_{N,r}^n(x_1, \dots, x_n) = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_{\tilde{S}_r^n} \rho_N^{n+k}(x_1, \dots, x_{n+k}) dx_{n+1} \cdots dx_{n+k}. \tag{10.5}$$

Combining (2.15) and (2.14) with (10.5) and the same equality as (10.5) for μ and applying the bounded convergence theorem, we obtain for each $r, n \in \mathbb{N}$

$$\sup_N \sup_{\tilde{S}_r^n} |m_{N,r}^n(x_1, \dots, x_n)| < \infty \tag{10.6}$$

$$\lim_{N \to \infty} m_{N,r}^n(x_1, \dots, x_n) = m_r^n(x_1, \dots, x_n) \quad \text{a.e..} \tag{10.7}$$

From this we see that the measures satisfy $\lim_{N\to\infty} \mu^N \circ \pi_{\tilde{S}_r}^{-1} = \mu \circ \pi_{\tilde{S}_r}^{-1}$ weakly in $\pi_{\tilde{S}_r}(\mathsf{S})$ for all r. Hence it only remains to prove that the sequence $\{\mu^N\}$ is tight in S.

Now we recall a closed subset S_0 in S is compact if and only if there exists an increasing sequence $\mathbf{a} = \{a_r\}_{r \in \mathbb{N}}$ of natural numbers such that $\sup_{\mathsf{s} \in \mathsf{S}_0} \mathsf{s}(\tilde{S}_r) \le a_r$ for all $r \in \mathbb{N}$ [19, Sect. 3.4]. Let $\mathsf{K}(r, a) = \{ \mathsf{s};\ \mathsf{s}(\tilde{S}_r) \le a \}$. Set $\mathsf{K}(\mathbf{a}) = \cap_{r \in \mathbb{N}} \mathsf{K}(r, a_r)$ for $\mathbf{a} = \{a_r\}_{r \in \mathbb{N}}$. Then we see that the set $\mathsf{K}(\mathbf{a})$ is compact in S because of the equivalence condition given above.

Let $\epsilon > 0$ be fixed. Note that $\pi_{\tilde{S}_r}(\mathsf{S})$ is also a Polish space because $\tilde{S}_r$ is Polish [19, Prop. 3.17]. Since $\{\mu^N \circ \pi_{\tilde{S}_r}^{-1}\}$ is tight as probability measures in $\pi_{\tilde{S}_r}(\mathsf{S})$, there exists a compact set K_r in $\pi_{\tilde{S}_r}(\mathsf{S})$ such that

$$\sup_N \mu^N \circ \pi_{\tilde{S}_r}^{-1}(\mathsf{K}_r^c) \le \epsilon 2^{-r}. \tag{10.8}$$

Moreover there exists an $a_r \in \mathbb{N}$ such that $\mathsf{K}_r \subset \mathsf{K}(r, a_r)$ because K_r is compact. We can and do take $a_r \in \mathbb{N}$ in such a way that $a_r < a_{r+1}$. By (10.8) and $\mathsf{K}_r \subset \mathsf{K}(r, a_r)$ we have $\sup_N \mu^N(\mathsf{K}(r, a_r)^c) \le \epsilon 2^{-r}$. Hence for $\mathbf{a} = \{a_r\}_{r\in\mathbb{N}}$ we have

$$\sup_N \mu^N(\mathsf{K}(\mathbf{a})^c) = \sup_N \mu^N(\bigcup_{r\in\mathbb{N}} \mathsf{K}(r, a_r)^c) \le \sup_N \sum_{r\in\mathbb{N}} \mu^N(\mathsf{K}(r, a_r)^c) \le \epsilon.$$

This implies $\{\mu^N\}$ is tight, which completes the proof. □

10.3 Proof of (8.31) and (8.32)

In this subsection we prove (8.31) and (8.32).

We begin with (8.31). Let $J_N(x) = I_N(x) - \frac{1}{2}\mathrm{sgn}(x)$. We note that $S_N(x)$ is an even function and $I_N(x)$, $D_N(x)$, and $J_N(x)$ are odd functions. By (8.4) for $S_N(x)$, $D_N(x)$, and $I_N(x)$ we see that

$$\begin{aligned} &\mathsf{K}^N_{\sin,1}(x)\mathsf{K}^N_{\sin,1}(-x) \\ &= \Theta(\begin{bmatrix} S_N(x) & D_N(x) \\ J_N(x) & S_N(x) \end{bmatrix} \begin{bmatrix} S_N(-x) & D_N(-x) \\ J_N(-x) & S_N(-x) \end{bmatrix}) \\ &= \Theta(\begin{bmatrix} S_N(x)^2 - D_N(x)J_N(x) & 0 \\ 0 & S_N(x)^2 - D_N(x)J_N(x) \end{bmatrix}. \end{aligned} \tag{10.9}$$

Hence by (8.2) and (8.29) we have $\mathcal{T}_1^N(x) = S_N(x)^2 - D_N(x)J_N(x)$ .This combined with (8.26)–(8.28) yields (8.31). We consider (8.32) next. By (8.6) for $S_N(x)$, $D_N(x)$, and $I_N(x)$ we see that

$$\begin{aligned} &\mathsf{K}^N_{\sin,4}(x)\mathsf{K}^N_{\sin,4}(-x) \\ &= \frac{1}{4}\Theta(\begin{bmatrix} S_N(2x) & D_N(2x) \\ I_N(2x) & S_N(2x) \end{bmatrix} \begin{bmatrix} S_N(-2x) & D_N(-2x) \\ I_N(-2x) & S_N(-2x) \end{bmatrix}) \\ &= \frac{1}{4}\Theta(\begin{bmatrix} S_N(2x)^2 - D_N(2x)I_N(2x) & 0 \\ 0 & S_N(2x)^2 - D_N(2x)I_N(2x) \end{bmatrix}. \end{aligned}$$

Hence by (8.2) and (8.29) we have $\mathcal{T}_4^N(x) = \frac{1}{4}(S_N(2x)^2 - D_N(2x)J_N(2x))$. This combined with (8.26)–(8.28) yields (8.32).

10.4 Proof of Lemma 9.2

The purpose of this subsection is to prove Lemma 9.2.

Let $\mathsf{g}(dz) = \frac{1}{\pi}\exp\{-|z|^2\}dz$ be the standard complex Gaussian measure. Let $\{\rho_N^n\}_{n\in\mathbb{N}}$ be the correlation function of μ^N_{gin} with respect to g. Then $\{\rho_N^n\}_{n\in\mathbb{N}}$ is given by

$$\rho_N^n(z_1, \ldots, z_n) = \det[K_N(z_i, z_j)]_{i,j=1,\ldots,n}, \tag{10.10}$$

where $K_N(z_1,z_2) = \sum_{k=0}^{N-1} \{z_1\bar{z}_2\}^k/k!$. We note that $\rho_N^n = \rho_{N,\mathrm{gin}}^n \pi^n e^{|z_1|^2+\cdots+|z_n|^2}$ and $K_N(w,z) = \pi e^{|w|^2/2} \mathsf{K}_{\mathrm{gin}}^N(w,z) e^{|z|^2/2}$ by construction. Let

$$K(z_1,z_2) = \sum_{k=0}^{\infty} \frac{\{z_1\bar{z}_2\}^k}{k!}, \quad K_N^*(z_1,z_2) = \sum_{k=N}^{\infty} \frac{\{z_1\bar{z}_2\}^k}{k!}. \tag{10.11}$$

Then $K = K_N + K_N^*$ by definition. Let

$$M_r^N = \int h_r(w)\overline{h_r(z)}\{|K(w,z)|^2 - |K_N(w,z)|^2 - |K_N^*(w,z)|^2\}\mathsf{g}(dw)\mathsf{g}(dz).$$

Lemma 10.1. *Let* $e_N^s = \sum_{k=0}^N s^k/k!$. *Then* $|M_r^N| \le 2\{1 - e^{-r^2} e_{N-1}^{r^2}\}\{1 - e^{-r^2} e_N^{r^2}\}$.

Proof. By $|K|^2 = |K_N|^2 + |K_N^*|^2 + K_N\overline{K}_N^* + K_N^*\overline{K}_N$ we have

$$\begin{aligned}
|M_r^N| &= |\int h_r(w)\overline{h_r}(z)\{K_N\overline{K}_N^* + K_N^*\overline{K}_N\}\mathsf{g}(dw)\mathsf{g}(dz)| \\
&= \frac{2}{(N-1)!N!}\{\int_{\tilde{S}_r} |w|^{2N-1}\mathsf{g}(dw)\}^2 \\
&\le 2\{\frac{1}{(N-1)!}\int_{\tilde{S}_r} |w|^{2N-2}\mathsf{g}(dw)\}\{\frac{1}{N!}\int_{\tilde{S}_r} |w|^{2N}\mathsf{g}(dw)\} \\
&= 2\{1 - e^{-r^2}e_{N-1}^{r^2}\}\{1 - e^{-r^2}e_N^{r^2}\}.
\end{aligned}$$

This completes the proof. □

We remark that the kernel K_N^* also generates the determinantal random point field denoted by μ_{gin}^{N*}. We write the variance with respect to μ_{gin}^{N*} as $\mathrm{Var}^{\mu_{\mathrm{gin}}^{N*}}$. A direct calculation shows the following.

Lemma 10.2. (1) *Let g be a bounded measurable function with compact support. Then*

$$\begin{aligned}
\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, g\rangle) = \int_{\mathbb{C}} |g(z)|^2 K_N(z,z)\mathsf{g}(dz) \\
- \int_{\mathbb{C}^2} g(w)\overline{g(z)}|K_N(w,z)|^2\mathsf{g}(dw)\mathsf{g}(dz)
\end{aligned} \tag{10.12}$$

$$\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, g\rangle) \le \frac{2}{\pi}\int_{\mathbb{C}} |g(z)|^2 dz. \tag{10.13}$$

(2) (10.12) *and* (10.13) *also hold for* μ_{gin}^{N*} *and* μ_{gin} *with* K_N *replaced by* K_N^* *and* K, *respectively.*

Proof. (10.12) is well known, so we omit the proof. Since $K_N(w,z)$ consists of a sum of pairs of orthonormal functions with respect to $\mathsf{g}(dz)$, we have the equality

$$K_N(z,z) = \int_{\mathbb{C}} |K_N(z,w)|^2 \mathsf{g}(dw). \tag{10.14}$$

Hence by (10.12) we obtain

$$\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, g\rangle) = \frac{1}{2}\int_{\mathbb{C}^2} |g(w) - g(z)|^2 |K_N(w,z)|^2 \mathsf{g}(dw)\mathsf{g}(dz). \tag{10.15}$$

Combining (10.14), (10.15), and the Schwartz inequality, and then using the estimates

$$0 \le K_N(z,z)(1/\pi)e^{-|z|^2} \le 1/\pi,$$

we conclude (10.13). The proof of (2) is the same as (1). □

Lemma 10.3 (Theorem 1.3 in [17])**.** $\sup_{1\le r} \frac{1}{r}\mathrm{Var}^{\mu_{\mathrm{gin}}}(\langle \mathsf{s}, h_r f\rangle) < \infty$.

Proof. This lemma is a special case of Theorem 1.3 in [17]. □

Lemma 10.4. $\sup_{1\le N}\sup_{1\le r} \frac{1}{r}\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r\rangle) < \infty$.

Proof. By $K = K_N + K_N^*$ and Lemma 10.2 we have

$$\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r\rangle) = \mathrm{Var}^{\mu_{\mathrm{gin}}}(\langle \mathsf{s}, h_r\rangle) - M_r^N - \mathrm{Var}^{\mu_{\mathrm{gin}}^{N*}}(\langle \mathsf{s}, h_r\rangle).$$

By Lemma 10.1 we have $|M_r^N| \le 2\{1 - e^{-r^2}e_{N-1}^{r^2}\}\{1 - e^{-r^2}e_N^{r^2}\}$. These, together with Lemma 10.3, complete the proof. □

Proof of Lemma 9.2. By $h_r f = z_0 h_r + h_r(f - z_0)$ and (10.13) we have

$$\begin{aligned}\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r f\rangle) &\le 2z_0^2\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r\rangle) + 2\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r(f - z_0)\rangle) \\ &\le 2z_0^2\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r\rangle) + \frac{4}{\pi}\int_{\tilde{S}_r} |h_r(f - z_0)|^2 dz.\end{aligned}$$

Therefore we obtain Lemma 9.2 by Lemma 10.4 and (9.3). □

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